Visualizing with the dynamic system Geogebra: the Fundamental Theorem of Algebra - TFA and its applications

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ABSTRACT: The relevance of Fundamental Theorem of Algebra - FTA is unquestionable. This fact provides implications in various research mathematical branches. Faced with such relevance, in this article, we discuss a particular method which involves a visual verification of what the FTA predicts and the idea related to the Rouché´s theorem. So, we present the use, in a complementarily way, of the dynamic system Geogebra and the CAS Maple. Our discussion emphasizes the graphics-geometry aspects. Thus, we provide the reader a heuristic exploration for future approaches in the context of teaching in Complex Visual Analysis.

KEYWORDS: TFA, Geogebra, Visualization, Complex Analysis - CA.

1. Introduction

The Fundamental Theorem of Algebra - FTA represents a great relevance and several applications in the study of Complex Analysis. In this article, we discuss some particular applications. In particular, we bring the Rouché´s Theorem and indicate visual interpretation from the Geogebra´s help. Moreover, we will indicate some important geometric ideas related with Argument Principle (BOTTAZZINI, 1986, p. 181).

Needham (2000) provides a differentiated approach to the teaching of Complex Analysis - CA. This author discusses the Rouché´s Theorem and them presents the FTA. Certainly, your argumentation is not the standard in CA. On the other hand, your approach is called Visual Complex Analysis. One of its goals is to show, indicate and describe qualitative properties related to the scientific concepts, by an intuitive and heuristic way.

Based on heuristic discussion, Needham (2000, p. 353) guides the reader to imagine “walking a dog round and round a tree in a park.” The closed path is developed around a tree. Needham considers leashes of variable length, “similar to a spring-loaded tape measure”. When we keep the leash short, the dog stays closer of your hell. Furthermore, “it is then clear that the dog is forced to walk round the tree the same number of times that you do. On another walk, thought, you decide to let out the leash somewhat so that the dog may scamper about, perhaps even running circles around you. Nevertheless, provided that you keep the leash short enough so that the dog cannot reach the tree, then again the dog must circle the tree the same number of times as you”.

This metaphorical description promotes the understanding of the classical formalism related to the Rouché´s theorem. However, Needham (2008, p. 353-354) uses the Rouché´s theorem for to demonstrate the TFA. In the next section, we will highlight certain elements of the historical order.

2. Historical context of the FTA

The FTA was been rigorously demonstrated over 200 years ago, in 1799, by C. F. Gauss at twenty years old, in this doctoral thesis (BOTTAZZINI, 1986). We enunciate the theorem that implies a strong property about ***C***[X] which establishes that the field of the complex numbers is algebraically closed. Kleiner (2007, p. 10) explain that “the theory of polynomial equation begun to emerge and amongst its main concern were the determination of the existence, nature and the number of roots of such equations.” Kleiner (2007, p. 12) still mentioned that there are several equivalents versions of the TFA. Here, we will consider the following statement.

Theorem 1: The field ***C*** of complex numbers if algebraically closed. Menini & Oystaeyen (2004, p. 463).

Certainly, we can mention other equivalents versions. In the historical point of view, the first formulation for this problem was made in the 17^{th}. Descartes, Girard, D´Alembert, Euler are mentioned in the specialized literature (BAHSMAKOVA & SMIRNOVA, 2000, p. 94-95) about their work around this specific mathematical problem (BOTTAZINI, 1986).

We still observe that TFA “provides a proof of the existence of complex roots of polynomial equation, but not a way to determine them explicitly.”(SHOKRANIAN, 2011,p.192). There is not an efficient mathematical method for determining all roots of a polynomial equation, either exact solutions or approximate solutions for such equation.

We developed a specific way to visualize and comprehend the existence of the roots. In this approach, we use the Dynamic system Geogebra and the CAS Maple. For example, we take the family: $f_1(z) = z$,

$$f_2(z) = z^2 + z + 1,\quad f_3(z) = z^3 + z^2 + z + 1,\quad f_4(z) = z^4 + z^3 + z^2 + z + 1,$$
$$f_5(z) = z^5 + z^4 + z^3 + z^2 + z + 1,\quad f_6(z) = z^6 + z^5 + z^4 + z^3 + z^2 + z + 1, \ldots,$$
$$f_n(z) = z^n + z^{n-1} + z^{n-2} + z^{n-3} + \cdots\cdots + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1.$$

In the figure 1, we show the graphic-geometric behavior of the plane curves that represents the level curves related the function $f_4(z)$ and $f_5(z)$.

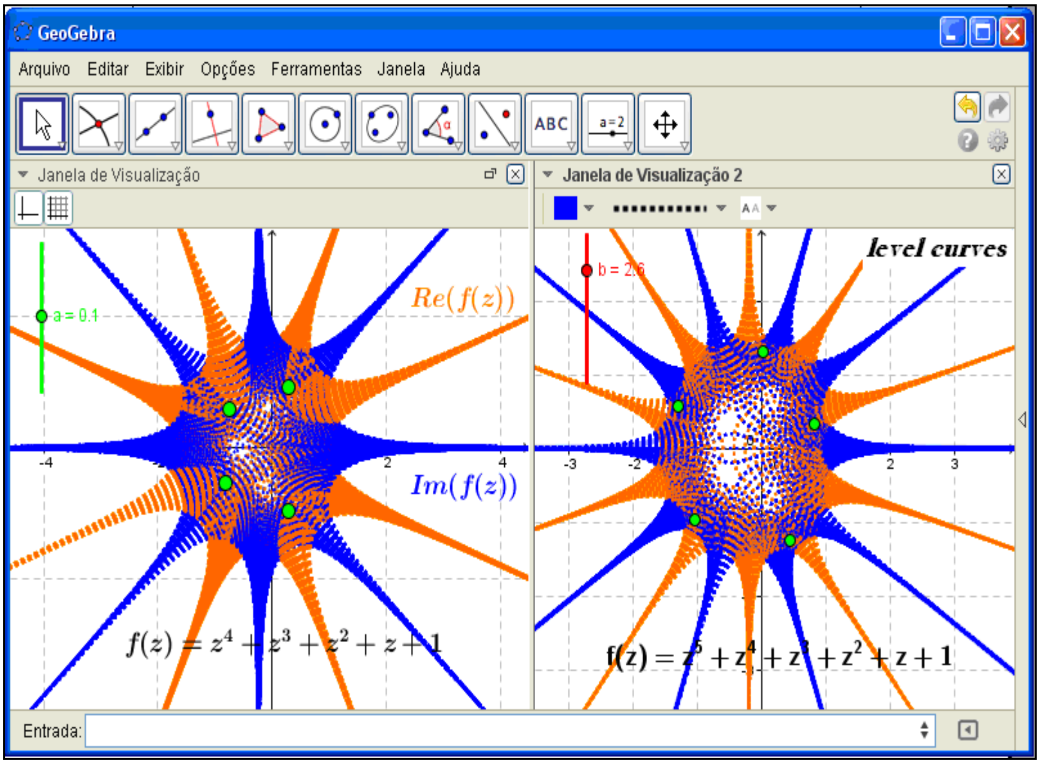

Figure 1. Visualization and understanding about the existence of roots by the TFA

We can understand the existence and the number of the roots related to the equations $f_4(z) = 0$ and $f_5(z) = 0$. Our techniques for describe visually the behavior of the roots involves the use of both software. With the CAS Maple, we can determine the expressions related to the real and imaginary parts, described in the following manner $f_4(z) = z^4 + z^3 + z^2 + z + 1 = \mathrm{Re}(f_4(z)) + i\,\mathrm{Im}(f_4(z))$. Then, with some basic commands of Geogebra, we insert dates for to describe the intersection of the level curves of the both parts above (fig. 1).

In Complex Variable study, traditionally, we chose some restriction to the range of the functions $f(z)$. In figure 2, we make the restriction $|z| \leq 1$ and observe, separately, the graphic behavior of $f(z) = z^9 - 2z^6 + z^2 - 8z - 2 = \mathrm{Re}(f(z)) + i\,\mathrm{Im}(f(z))$, for example.

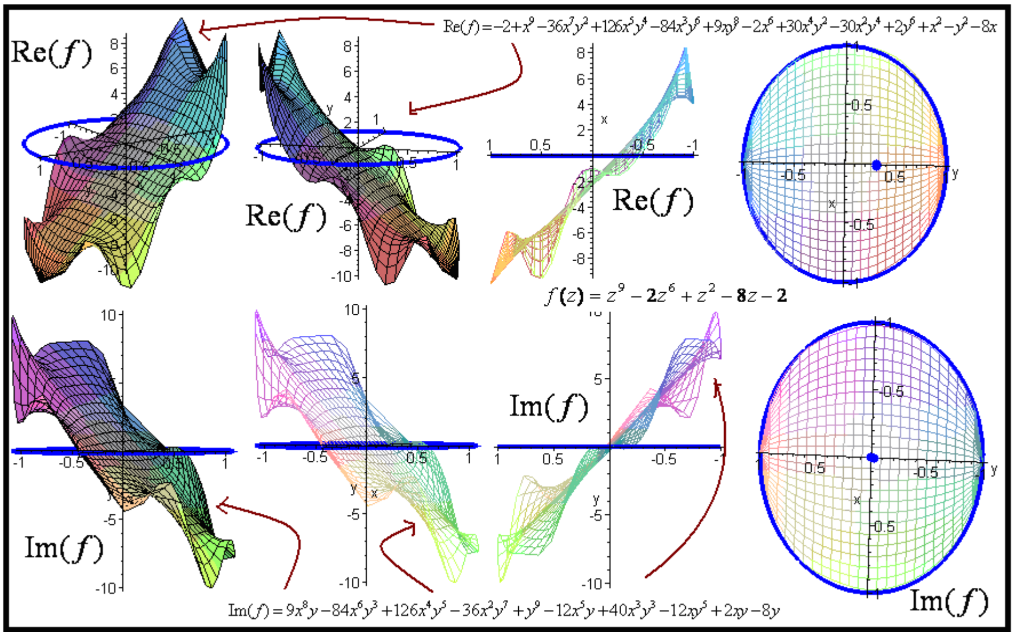

Figure 2. Description of the real and imaginary parts of the complex variable function $f(z)$ in the space IR^3 with the restriction in the disk $|z| \leq 1$

In the graphic determined by $\mathrm{Re}(f)$, we can observe in the space IR^3 that exist only one intersection point across the axes. Similarly, in the graphic determined by $\mathrm{Im}(f)$, we can observe too that exist only one intersection point, with the restriction to the unitary disc indicated by $|z| \leq 1$.

On the other hand, we can take other restrictions, like $|z| \leq 2$ or the ring $1 \leq |z| \leq 2$. In the figure 3, we can explore the graphs and understanding that the number of the intersection with the xOy plane increased.

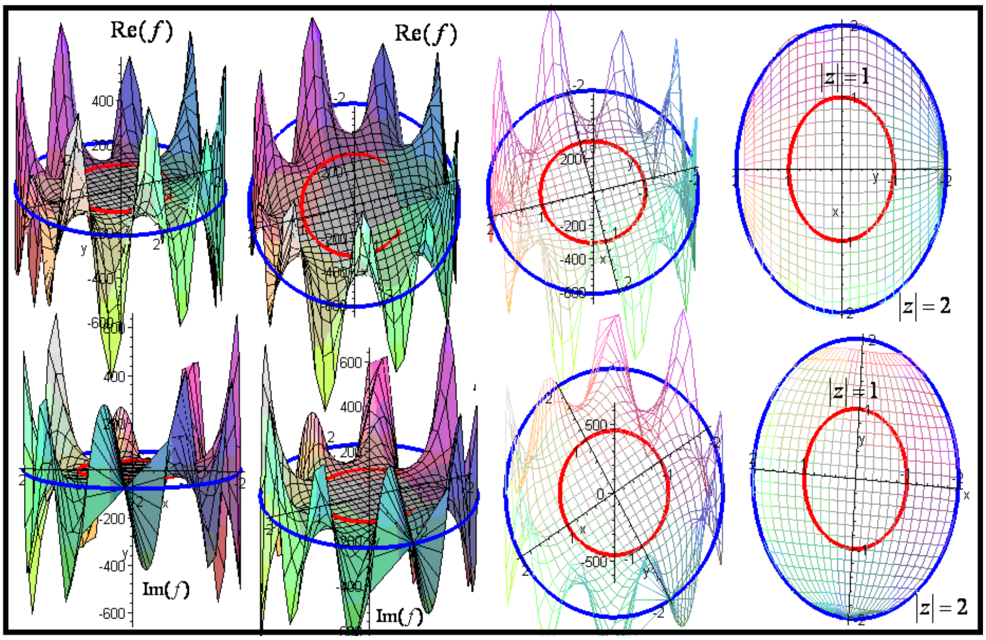

Figure 3. Description of the real and imaginary parts of the complex variable function $f(z)$ in the space IR^3 with the restriction in the disk $|z| \leq 2$

We can compare the behavior in the two cases. In the figures 2 and 3, we can apply a techniques presented by Needham (2000), with the intention to study the behavior graph in the complex variable. In this sense, Needham (2000, p. 56) comments that "the image $f(z)$ of a point z may be described by its distance $|f(z)|$ from the origin, and the angle $\arg[f(z)]$ it makes with real axes". From this consideration, we note that $|f(z)| = 0 \leftrightarrow f(z) = 0$.

Now we have a way to compare the graphic-geometric information extracted from 3D representation with others from the 2D representations. In space, we can visualize the surfaces related to the real and imaginary parts of each function in the complex variable. From the figure 3, we can conjecture that the other eight roots of $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ are in the ring described for the analytical condition $1 \leq |z| \leq 2$.

Certainly, this visual method not provides the all information that we need to confirm our conjecture. On the other hand, by means of the graphics that we exhibited (fig. 1), with the Geogebra´s help, we acquire the numerical feeling about the existence of the equation polynomial roots.

With special attention in view of the preliminary understanding of the students, we strongly suggest the exploitation of technological means, coupled with logical formulations pertaining to Mathematics

Before starting the next section, we recall that the book *Cours D´Analyse* is devoted to Complex Analysis. Bottazinni (1986, p. 126) explains that "constitutes one of Cauchy´s work most fundamental contributions to Mathematics, and probably his most important.". This author also points out that the mysterious calculation around the complex number disappears and the handling of imaginary quantities becomes indispensable in the mathematical research (ALVES, 2012).

There are several ways to demonstrate the TFA. For example, we can apply the Louiville´s theorem for this (LINS, 1993, p. 199). Moreover, we show that the proof of this theorem, in other ways, allows the exploration and the design of some constructions in the software that can enrich our imagination and stimulate our private perception. Certainly, an indispensable mental ability in mathematical research!

We recall, for example, the metaphorical description provided by Needham (2000, p. 353) in the introduction of this work. The concern of this author, in view of intuitive transmission of certain mathematical ideas, will continue to affect our perspective. In the next section, we will use the following symbol $\nu[f(\Gamma),0]$ which indicates the winding number of a closed loop L about the origin 0 (NEEDHAM, 2000, p. 338).

3. Visualization of Dynamic System Geogebra: the case of Rouché´s theorem and the notion of winding number

In the introductory section, we presented a heuristic approach suggested by Needham (2000) related to a mathematical theorem. In order to obtain a geometry and intuitive interpretation of it, this author relies on metaphors. In fact, "let a tree be the origin of *C*, and let your walk be the image path traced by $f(z)$ as z traverses a simple loop Γ. Also, let the complex number from you to the dog be $g(z)$, so the dog´s position is $f(z)+g(z)$.". Needham (2000, p. 353) records too the essential condition that $|g(z)|<|f(z)|$.

The last inequality is the requirement that the leash not stretch to the tree on Γ. Needham (2000, p. 354) explain that "the Argument Principle then inform us that: se $|g(z)|<|f(z)|$ on Γ, then $f+g$ must have the same number of zeros inside Γ as f.".

From this characterization, we can prove the TFA. Indeed, lets consider $p(z)=z^n+Az^{n-1}+Bz^{n-2}+\cdots+E$, z∈*C*. We want to prove the theorem 1. However, before implementing such formal verification, we make some observations about polynomial functions and the Argument Principle – AP. In fact, Needham (2000, p. 344) provides the reader a figure that involves the Argument Principle´ idea. He considers a circle indicated by Γ and the image of the function $f(\Gamma)$, (see figure 4).

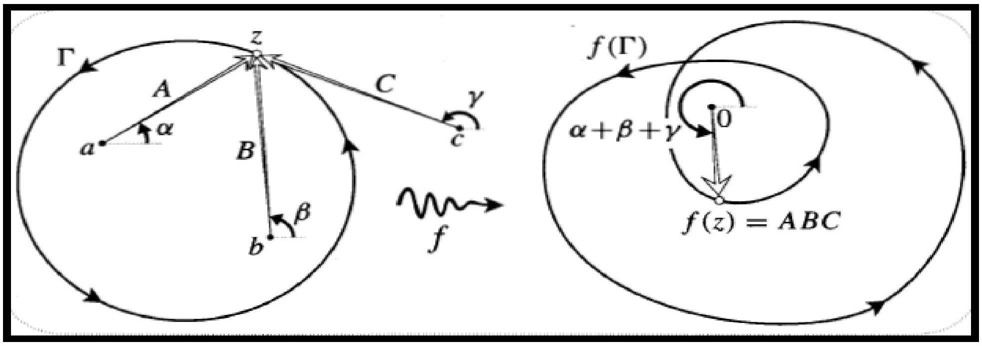

Figure 4. Needham (2000, p. 345) provides a static figure for describe the AP

The author states that "notice that Γ encircle two zeros of the mapping, while $f(\Gamma)$ has a winding number of 2 about zero. This is no accident. Since angles add when we multiply complex numbers, the number of revolutions executed by ABC is just the sum of revolutions executed separately by each of A, B and C.". Moreover, based on the figure 4, He writes that $\nu[f(\Gamma),0]=2$. In the next figure, we show a similar construction, with the following statement $\nu[f(\Gamma),0]=3$.

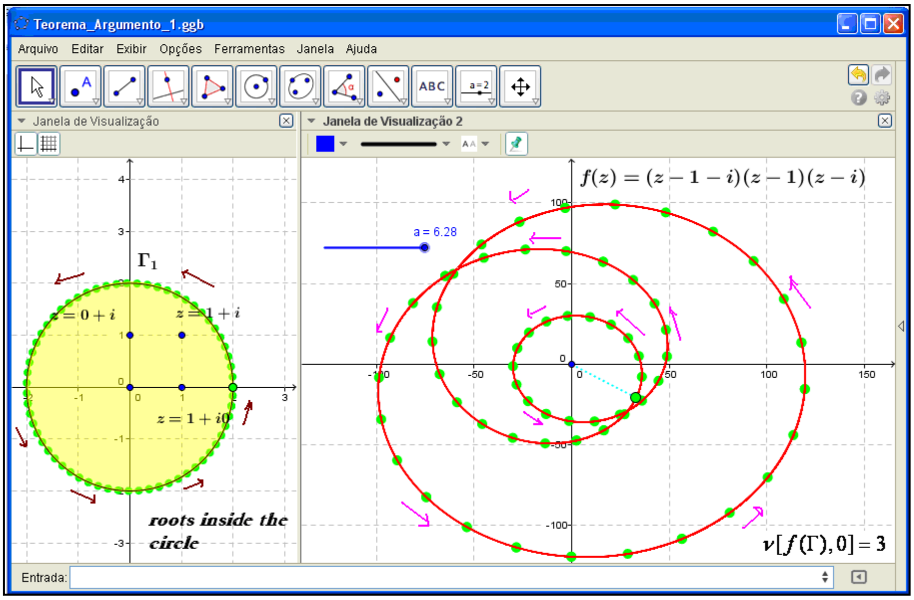

Figure 5. Dynamic description of the idea related to the AP with the Geogebra´ help

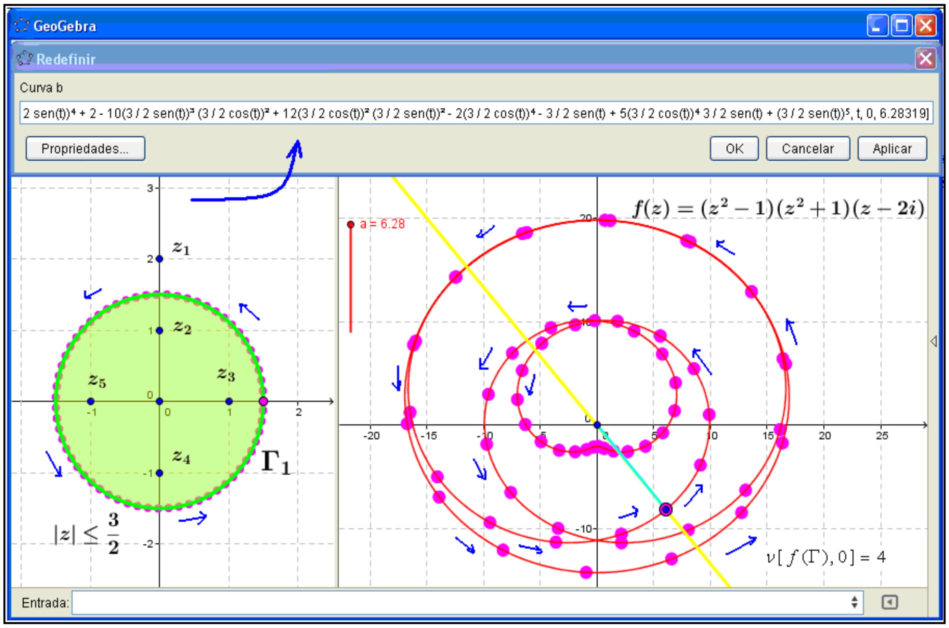

Figure 6. Dynamic description of the idea related to the AP with the Geogebra´ help

In the figure 5, we considered $f(z)=(z-1)(z-i)(z-1-i)$. We take the a disk, described for $|z|\leq 2$ Γ. So, Γ_1 encircles the three roots $1, i, 1+i$ of $f(z)$. On the right side, when we consider the following parameterization for the circle $\alpha(t)=(2\cos(t), 2sen(t))$, we can describe graphically the $f(\Gamma)$. However, with the CAS Maple, we determine the algebraic expressions related to the real and imaginary parts which we indicated by $\mathrm{Re}(f(x+iy))$ and $\mathrm{Im}(f(x+iy))$ in the tridimensional space.

Well, this idea is closely related to the proof of TFA. Needham (2000, p. 354) employs the Rouché´s theorem for demonstrate the TFA. He took the polynomials $f(z)=z^n$ and $g(z)=Az^{n-1}+\cdots+E$, where $p(z)=z^n+Az^{n-1}+Bz^{n-2}+\cdots+E=f(z)+g(z)$. We must choose a circle Γ defined by $|z|=1+|A|+|B|+\cdots+|E|$. Finally, Needham (2000, p. 354) concludes "since f has n roots inside Γ (all at the origin), Rouché says that $p(z)$ must too. Now, we will illustrate some situations involving the use of this important theorem.

Let's consider the function $f(z)=z^9-2z^6+z^2-8z-2$. With the use of CAS Maple, we obtain some complicated expressions:

$$\mathrm{Re}(f)=-2+x^9-36x^7y^2+126x^5y^4-84x^3y^6+9xy^8-2x^6+30x^4y^2-30x^2y^4+2y^6+x^2-y^2-8x$$

$\text{Im}(f) = 9x^8y - 84x^6y^3 + 126x^4y^5 - 36x^2y^7 + y^9 - 12x^5y + 40x^3y^3 - 12xy^5 + 2xy - 8y$

We take these expressions and obtain the level curves in the figure below. We show red and blue curves. Geometrically, these curves describe the level curves in the plane, associated to the surface indicated by $\text{Re}(f(x+iy)) = u(x,y)$ and $\text{Im}(f(x+iy)) = v(x,y)$. We know the basic properties from the graph $Graf(f(z)) = (x, y, u(x,y), v(x,y)) \subset IR^4$.

In this case, we take the functions $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ and ask how many zeros does f have in $|z| < 1$. We choose compare f and $-8z$. By the Rouch´s theorem, we write analytically:

$$|f(z) + 8z| = |z^9 - 2z^6 + z^2 - 2| \leq |z^9| + |2z^6| + |z^2| + |2| \leq 1 + 2 + 1 + 2 = 6 \leq 8|z|.$$

So, by the Rouché`s theorem, we conclude that $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ and $8z$ have the same number of zeros in $|z| < 1$. Indeed, we see, on the right side of the figure 7, a single root. Thus, we visually relate the region, indicated in the figure 7 by Γ_1, with the expected numbers of the roots. On the other hand, in virtue of TFA, this polynomial has exactly nine roots!

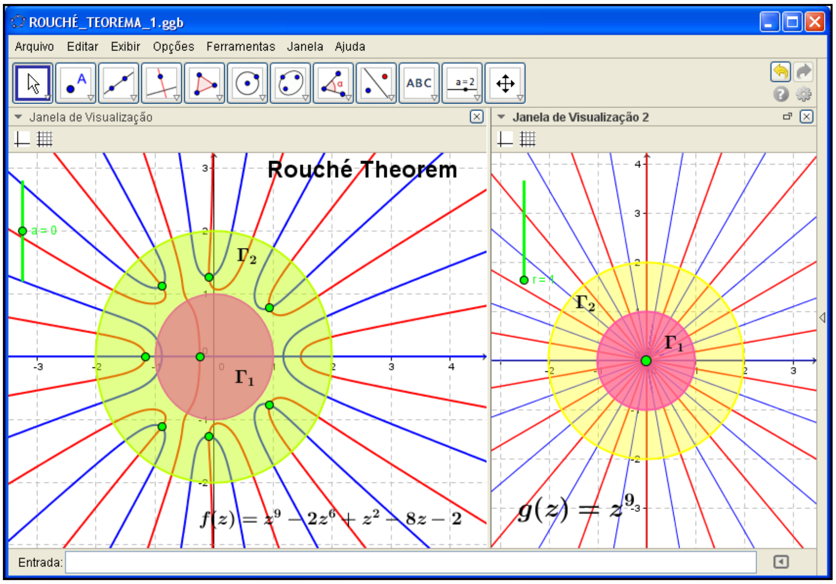

Figure 7. Visualizing the region on the plane and localization of the roots of polynomial equation of degree nine

Consider now the condition $|z| < 2$. We observe that we located only one of the zeros (fig. 7). On the other hand, by the FTA, we must have precisely nine zeros (in Γ_1). In this case, we compare $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ and z^9. So, we write the analytically condition: $\left| f(z) - z^9 \right| = \left| -2z^6 + z^2 - 2 \right| \le 128 + 4 + 16 + 2 = 150 < |z|^9 < |z|^9 + |f(z)|$

So, en virtue the Rouché´s theorem, we can assure that $f(z) = z^9 - 2z^6 + z^2 - 8z - 2$ and $g(z) = z^9$ has the same numbers of the roots in the $|z| < 2$. However, we had already indicated one of these. This new information indicates that the remaining roots are in the ring $1 \le |z| \le 2$.

We take $f(z) = 2z^5 - z^3 + 3z^2 - z$ and $g(z) = -8$. We obtain that $|f(z) + g(z)| = |2z^5 - z^3 + 3z^2 - z| \le 2 + 1 + 3 + 1 = 8 \le |g(z)| + |f(z)|$. Due the Rouché´s theorem, the functions $f(z) = 2z^5 - z^3 + 3z^2 - z$ and $g(z) = -8$ has the same numbers of zeros in the unit circle. So, the equation has no zeros in this unite circle Γ_1. We see this in the figure 8, on the right side.

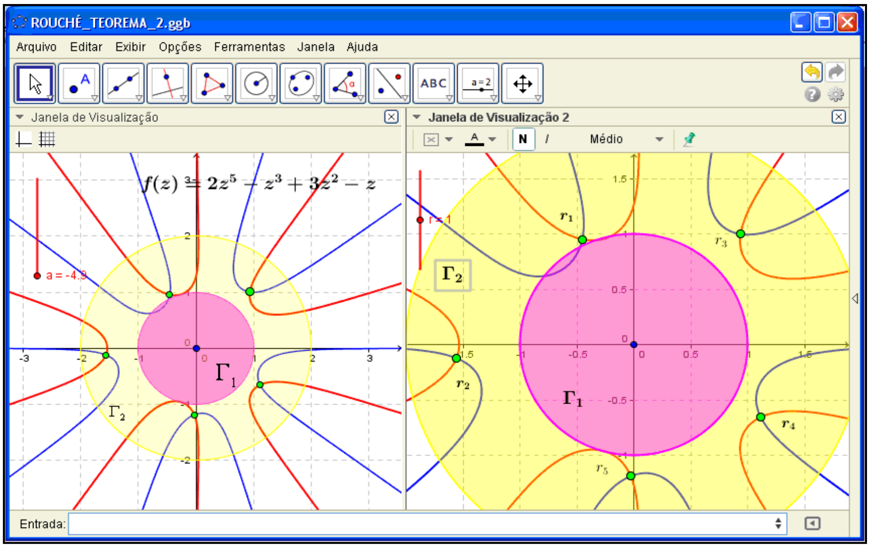

Figure 8. With the software Geogebra, we can visually confirm the dates provided by the Rouché´s theorem and the localization of the roots

In the first view, we did not differentiate the relative position between the root and the disc $|z| \le 1$ (Γ_1) (see figure 8). Thus, we used a software function with the purpose to identify its actual position on the disc. Visually, we conclude that all five roots are in the ring $1 \le |z| \le 2$ (on the right side).

Henceforth, we shall analyze the analytical point of view the location of the roots of this polynomial function. We will consider now the other functions $f(z) = 2z^5 - z^3 + 3z^2 - z$ and $g(z) = -2z^5$ and indicate:

$$|f(z)+g(z)| = |-z^3+3z^2-z| \le 1+3+1 = 5 \le 2|z|^5 = |g(z)| \le |f(z)|+|g(z)|.$$

So, we conclude that $|f(z)+g(z)| \le |f(z)|+|g(z)|$. By the Rouché´s theorem, functions $f(z) = 2z^5 - z^3 + 3z^2 - z$ and $g(z) = -2z^5$ has the same numbers of zeros in the ring $1 \le |z| \le 2$. We confirm our preliminary conjecture based on the perception. Our approach allows an interpretation supported by the visualization of the Rouché´s theorem in the plane.

4. Final remarks

We discussed in this article some relevant and fundamentals mathematical theorems that have innumerous applications and implications in the Complex Analysis. In the specialized literature on the History of Mathematics, we acquired an understanding about the importance and the problems that led to the establishment of TFA and the Rouché´s theorem.

We explored a heuristic approach and a dynamic interpretation for these theorems and some ideas related to the Argument Principle - AP. However, the use of the Geogebra and the CAS Maple based a complementary perspective, allows us to obtain interesting data that go beyond the formal-logical meanings from the conditional standard statements.

About it with the software, we can inspect the area of the plane in which we determine the number of roots of a polynomial (see figures 6 and 7). Furthermore, we acquire an understanding of the topological location of roots. Finally, we can prove, from a visual standpoint, the property provided by the Rouché´s theorem. Indeed, we can adjust the disk and visualizing the ring in which we expect to find all or some of the polynomial roots.

Concerning the TFA, we found the authors that suggest certain limitations of the theorem (SHOKRANIAN, 2011; MENINI & VAN OYSTAEYEN, 2004), while recognizing its great historic merit. In fact, this theorem provides a way to prove the existence of roots of a complex polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$, but not a way to determine them explicitly.

From the mathematical point of view there are several ways to show the TFA. However, in this article, we emphasized its statement from the Rouché´s theorem. Although, one of the most elegant applications of Liouvile´s theorem is precisely the TFA (KRANTZ, 2007, p. 97). Thus, using the software, we can confront and compare the visual data with the formal and logical arguments, traditionally employed in the books of CA.

Finally, we indicated several constructions with dynamic system Geogebra that allows the description, in the graphical point of view, some geometric and topological notions related to the Principle of the Argument. We evidenced, for example, the concept involving the number of turns of a curve around the origin O (winding number). In the previous sections, we quickly indicated the symbol $\nu[f(\Gamma),0]$. However, to perform a direct count of the number of turns of the curve with self-intersection can become a difficult task (NEEDHAM, 2000, p. 341). Although, this notion is needed for comprehend several properties in the theory of integration. In future work, we will discuss this specific issue in detail, always keeping a special attention in view of the teaching (ALVES, 2012; 2013b; 2013b).

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