# Discovering and Extending Viviani's Theorem with GeoGebra

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**ABSTRACT:** In this paper I illustrate how we can use GeoGebra to guide learners in the processes of discovering and extending Viviani's Theorem. First, the theorem is formulated as an open-ended task so that students can discover the theorem on their own. Second, the theorem is extended to exterior points using the concept of signed distance. Third, the theorem is modified to extend it to sums of particular lengths of segments and areas. Finally, proofs of the conjectures are also included to provide learners a complete learning mathematical experience.

**KEYWORDS:** Viviani's theorem, GeoGebra, discovery, extending, exterior points, lengths of segments, areas, proof.

## 1. Introduction

One of the most motivating features of using Dynamic Geometry, such as GeoGebra (GG), in teaching and learning mathematics is its capability to facilitate the processes of discovering and extending mathematical theorems. In this paper, I outline how we, teachers and teacher educators, can use GG to discover and extend one of the most elegant, fruitful, and simplest theorems of modern Euclidean geometry: Viviani's theorem.

## 2. Discovering Viviani's theorem

To provide students with opportunities to discover Viviani's theorem, I usually reformulate it as a problem:

Let P be a point in the interior of an equilateral triangle ABC. Let D, E, and F be the feet of the perpendicular segments from P to each of its sides (Fig. 1). a) What property does PD + PE + PF have? b) Interpret geometrically the sum of the distances from P to the sides of the triangle, and c) Extend the pattern to other points of the plane.

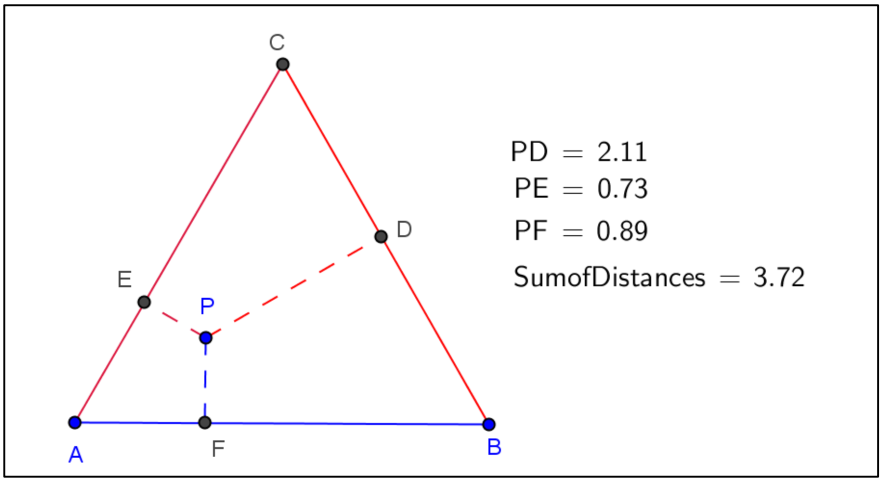


**Figure 1**: Representation of Viviani's problem

Figure 1 displays a graphical representation of the problem. To investigate what property the sum of the distances from P to the sides of ΔABC (PD + PE + PF) has, we drag point P in the interior of the triangle and observe its effect on PD + PE + PF (SumofDistances). Most students are delighted to discover that PD + PE + PF is invariant for specific equilateral triangles (Fig. 2). That is, PD + PE + PF is a constant. Further dragging provides additional empirical evidence that PD + PE + PF seems to be a constant for any equilateral triangle.

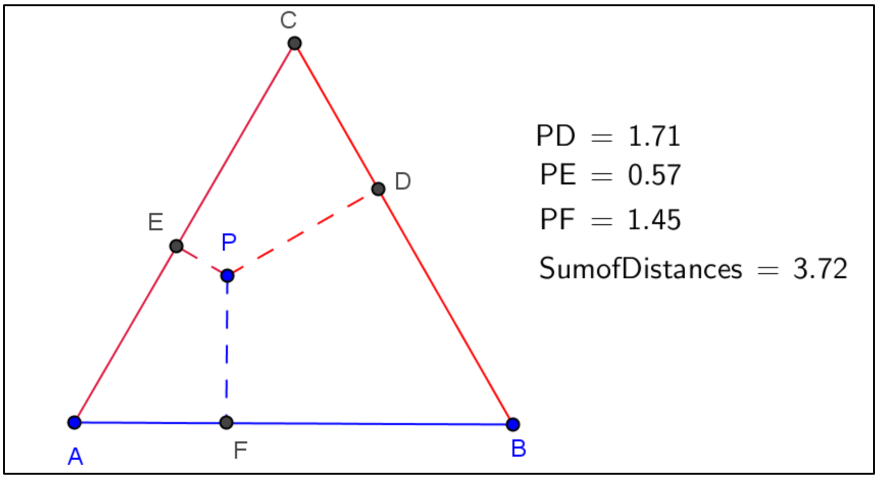


**Figure 2**: PD + PE + PF is invariant

To help students discover the geometric interpretation of PD + PE + PF, I often suggest to them to drag P to an extreme position, as suggested by Polya (1945). In this case, extreme points are points on the sides of the triangle or, better still, any of the three vertices of the triangle. Figure 3 shows the situation when P is very close to vertex C. Again, it is amazing to discover that PD + PE + PF has a simple and elegant geometric interpretation: it is the length of the altitude of the equilateral triangle ABC.

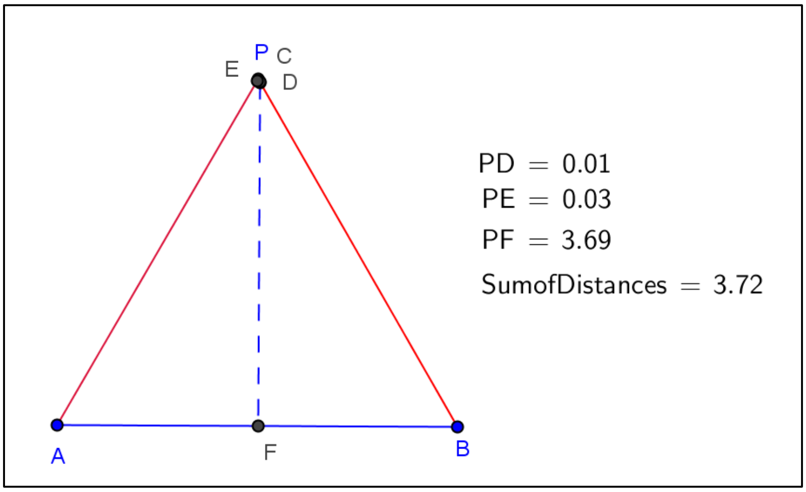


**Figure 3**: P is an extreme point

The use of GG is instrumental in helping learners discover mathematical patterns and invariants. Even though GG, or any other dynamic geometry software, provides a very high level of conviction regarding the veracity of a conjecture, I strongly believe that we, teachers and students, should develop mathematical proofs. A mathematical proof allows us not only to verify that a conjecture is true, but also to gain a better understanding of *why* the conjecture is true. Having a deeper understanding of why a result is valid can, in turn, lead to a whole series of new discoveries, extensions, and generalizations.

To gain insight into why the sum of the distances from any point in the interior of an equilateral triangle to its sides is the length of its altitude, I guide my students in the construction of a proof. First, I suggest to them to construct segments $\overline{PA}$, $\overline{PB}$, and $\overline{PC}$ (Fig. 4). Second, I ask them what, besides PD + PE + PF, remains invariant as we drag point P to several locations in the interior of the equilateral triangle ABC. A few students usually notice that the sum of the areas of the three triangles does not change as we move P to several locations in the interior of ΔABC. We use this invariant property to develop the following proof for our conjectures: Let $s$ = AB. It follows now that Area (ΔABC) = Area (ΔBCP) + Area (ΔACP) + Area (ΔABP) = $\frac{s(PD)+s(PE)+s(PF)}{2} = \frac{s}{2}(PD + PE + PF)$. Because Area (ΔABC) = $\frac{sh}{2}$, we obtain that $h$ = PD + PE + PF. In other words, PD + PE + PF is a constant and equals the height of the triangle.

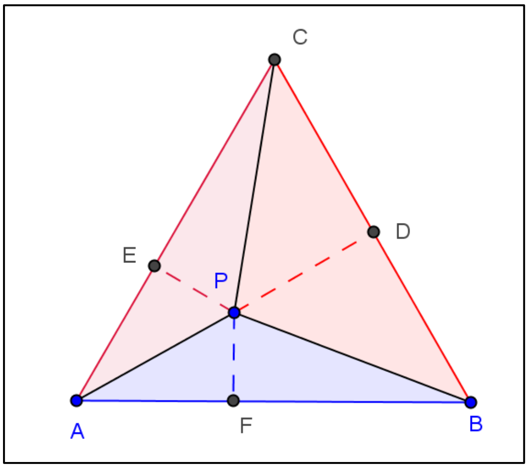


**Figure 4**: Diagram used to prove Viviani's theorem

Our next task is to investigate whether PD + PE + PF is invariant for other points of the plane besides the interior points of an equilateral triangle. Some students quickly notice both empirically and extending the previous proof that this quantity is also invariant for points on the sides of the triangle. Putting all of these observations together leads to the statement of Viviani's theorem:

> *Viviani's theorem*: Let P be a point in the interior or on one of the sides of an equilateral triangle. The sum of the distances from P to the sides of the triangle equals the length of its altitude.

It seems that by now we have completed the investigation of Viviani's theorem. However, nothing could be further from the truth. As suggested by Polya (1945), solving problems involves looking back at a solved problem to find out whether a particular pattern or solution extends to other situations. At this point we then should consider posing "what-if" questions (Brown &

Walter, 1990). One of the questions that students usually pose is: what happens if point P is outside of the triangle? (Fig. 5).

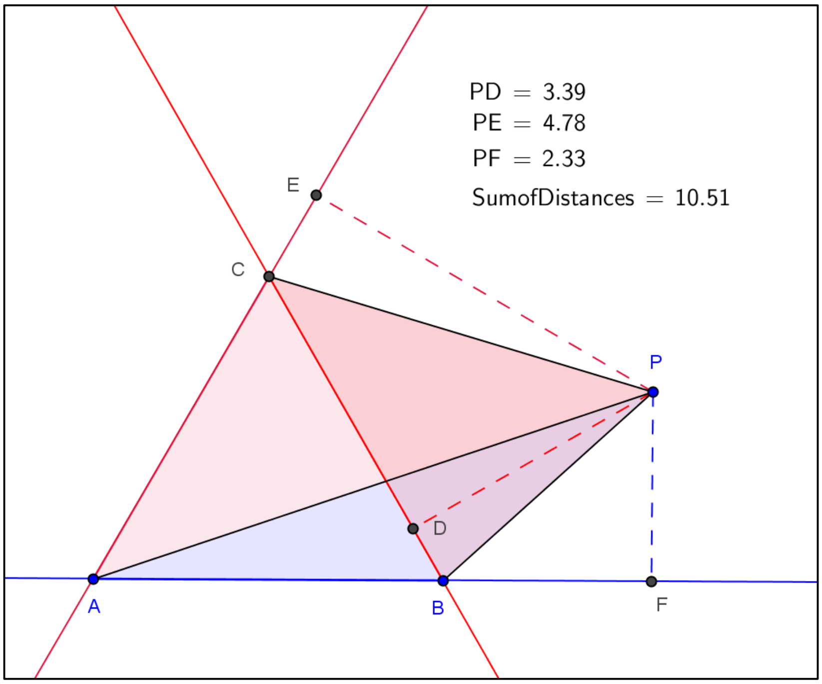


**Figure 5**: What happens if P is an exterior point?

## 3. Extending Viviani's theorem to exterior points

Because the new relationship is not obvious, I ask students to look back at the relationship of the areas between triangle ABC and triangles ABP, BCP, and ACP (Area (ΔABC) = Area (ΔABP) + Area (ΔBCP) + Area (ΔACP)). At this point students immediately realize that Area (ΔABC) = Area (ΔABP) + Area (ΔACP) – Area (ΔBCP) from where we obtain the relationship $\frac{sh}{2} = \frac{s}{2}(PF + PE - PD)$ or $h$ = PE + PF – PD. Some students verify this last relationship with GG (Fig. 6). In this case the expression PEpPFmPD computes PE + PF – PD (PE plus PF minus PD).

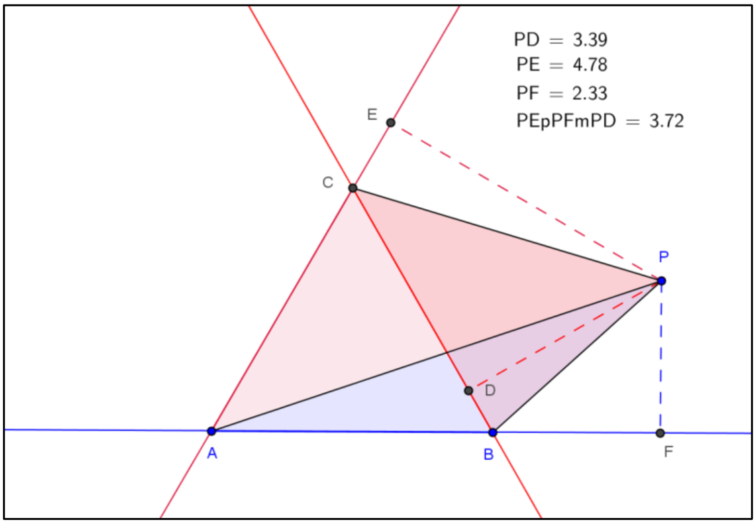


**Figure 6**: $h$ = PE + PF – PD

We next tackle the case illustrated in figure 7. Students immediately visualize that Area (ΔABC) = Area (ΔABP) – Area (ΔBCP) – Area (ΔACP), which leads to the conclusion that $h$ = PF – PE – PD. The other four cases produce, respectively, $h$ = PD + PF – PE, $h$ = PD – PE – PF, $h$ = PD + PE – PF, and $h$ = PE – PD – PF.

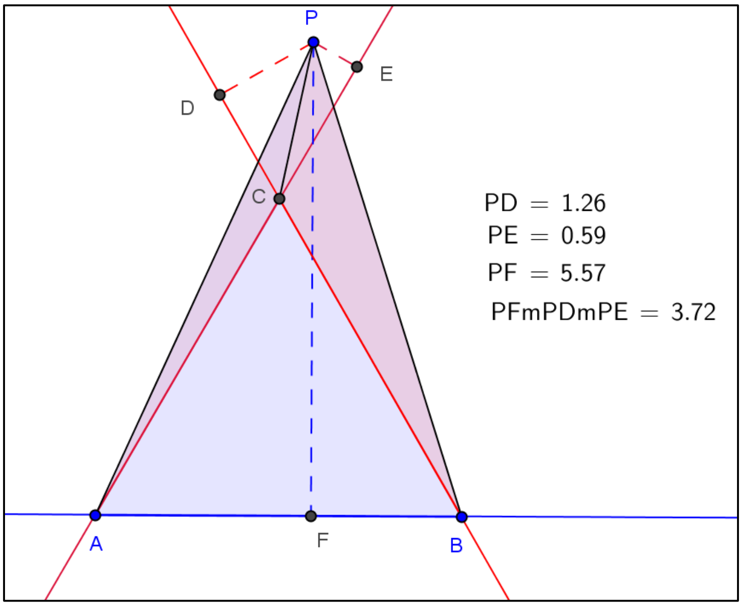


**Figure 7**: $h$ = PF – PD – PE

One of the ubiquitous features of mathematics is its endless search to unify different cases of a theorem. To this end, we can use the notion of directed distances to unify the seven cases for the position of P. We say that a distance from a point P located in the exterior of a triangle ABC to one of its sides is positive if the point and triangle are on the same half plane determined by the side in question. If the point and triangle ABC are on different half planes, then the distance is negative. Thus, in figure 6 both distances PE and PF are positive while distance PD is negative. Similarly, in figure 7 distance PF is positive while both distances PD and PE are negative. We can then state the following theorem.

> *Generalized Vivianis's theorem*: The sum of the signed distances from any point in the plane of an equilateral triangle to its three sides equals the length of its altitude.

## 4. Extending Viviani's theorem to segments on the sides of the equilateral triangle

As argued by Movshovitz-Hadar (1988), almost every mathematical theorem is an endless source of surprise. Viviani's theorem is certainly not an exception. By asking and investigating "what-if" questions on our own on a regular basis until it becomes a spontaneous act, all of us, teachers and learners, have the potential to discover a new conjecture, at least new to the discoverer or learner.

To provide learners with richer experiences in this domain using Viviani's theorem, we should ask them to examine segments $\overline{AE}$, $\overline{CD}$, and $\overline{BF}$, as displayed in figure 8a, and make a conjecture about them. Initially only a few students may notice the pattern but after suggesting considering AE + CD + BF, almost all of them may conjecture with the help of GG that such a sum is a constant and a few may further notice that the constant is the semiperimeter of ΔABC (Fig. 8b). Notice that AEpCDpBF in figure 8b represents AE + CD + BF. Further dragging of point P in the interior of the triangle provides additional evidence to support the conjecture.

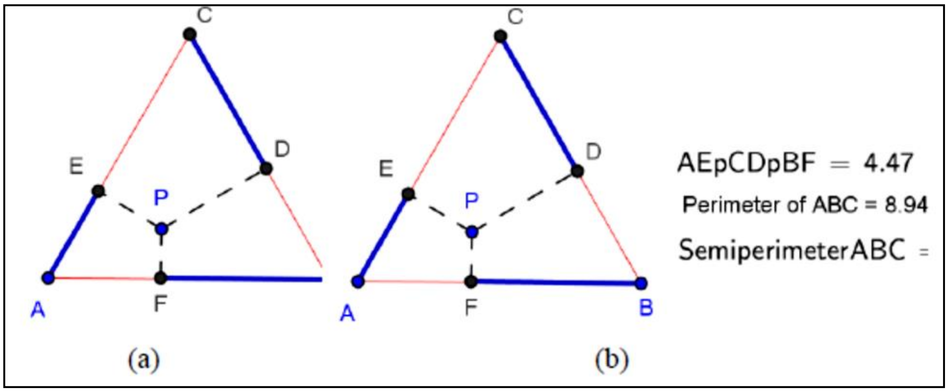


**Figure 8**: AE + CD + BF is invariant

It is worth mentioning that this conjecture was originally discovered by Duncan Clough, a high school student, while he was exploring Viviani's theorem using Dynamic Geometry (De Villiers, 2013). For those students and readers who love algebraic proofs, I provide the following argument based on coordinate geometry. I should mention that GG was very helpful in verifying intermediate steps and finding a couple of mistakes that I made in the initial argument.

Place the equilateral triangle as shown in figure 9 and assign coordinates to its vertices A and B as shown. The coordinates of point C are:

$x = \frac{a}{2}$ and $y = \frac{a\sqrt{3}}{2}$.

Let P be a point located in the interior or on any side of ΔABC. Let D, E, and F be the feet of the perpendicular segments constructed from P to each of the sides of the triangle (see figure 9). Assign coordinates to points P, D, E, and F as indicated in figure 9.

We then have slope of $\overline{AC} = \sqrt{3}$, slope of $\overline{BC} = -\sqrt{3}$, and slope of $\overline{AB} = 0$. From here it follows that slope of $\overline{PD} = \frac{1}{\sqrt{3}}$ and slope of $\overline{PE} = -\frac{1}{\sqrt{3}}$. Using the fact that $\frac{v-c}{u-b} = -\frac{1}{\sqrt{3}}$ we obtain

$$\frac{b}{\sqrt{3}} + c = \frac{u}{\sqrt{3}} + v \qquad (1)$$

On the other hand,

$$c = \sqrt{3}\, b \qquad (2)$$

Substituting (2) in (1) results in $b = \frac{u+\sqrt{3}v}{4}$ and $c = \frac{\sqrt{3}u+3v}{4}$. Similarly, the relationships between $(d, e)$ and $(u, v)$ can be represented by the following system of equations:

$$e - \frac{d}{\sqrt{3}} = v - \frac{u}{\sqrt{3}} \qquad (3)$$

$$e + \sqrt{3}d = \sqrt{3}a \qquad (4)$$

Solving (3) and (4) yields $d = \frac{u-\sqrt{3}v+3a}{4}$ and $e = \frac{3v-\sqrt{3}u+\sqrt{3}a}{4}$.

Next, we compute AE, CD, and BF.

$$\text{AE} = \sqrt{\left(\frac{u+\sqrt{3}v}{4}\right)^2 + \left(\frac{\sqrt{3}u+3v}{4}\right)^2} = \sqrt{\frac{4u^2+8\sqrt{3}uv+12v^2}{16}} =$$

$$\frac{\sqrt{u^2+2\sqrt{3}uv+3v^2}}{2} = \frac{u+\sqrt{3}v}{2}$$

$$\text{CD} = \sqrt{\left(\frac{u-\sqrt{3}v+3a}{4} - \frac{a}{2}\right)^2 + \left(\frac{3v-\sqrt{3}u+\sqrt{3}a}{4} - \frac{\sqrt{3}a}{2}\right)^2} =$$

$$\sqrt{\frac{4u^2+12v^2+4a^2-8\sqrt{3}uv+8au-8\sqrt{3}av}{16}} = \sqrt{\frac{u^2+3v^2+a^2-2\sqrt{3}uv+2au-2\sqrt{3}av}{4}}$$

$$= \frac{u-\sqrt{3}v+a}{2}$$

$\text{BF} = a - u$

Thus,

$\text{AE} + \text{CD} + \text{BF} = \frac{3a}{2}$, which is the semi-perimeter of ΔABC.

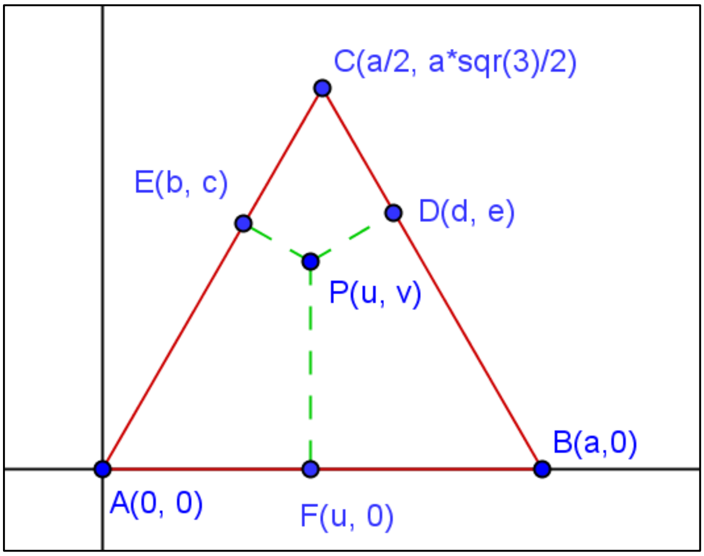


**Figure 9**: Diagram for coordinate proof

One habit of mind that we should help students develop is a disposition toward solving a problem using a multitude of strategies. The existence of right triangles in the geometric configuration suggests using the Pythagorean Theorem. Labeling the geometric objects as shown in figure 10 results in the following equalities

$$u^2 = m^2 - PE^2 \qquad (1)$$

$$v^2 = l^2 - PD^2 \qquad (2)$$

$$w^2 = n^2 - PF^2 \qquad (3)$$

$$a^2 - 2au + u^2 = l^2 - PE^2 \qquad (4)$$

$$a^2 - 2av + v^2 = n^2 - PD^2 \qquad (5)$$

$$a^2 - 2aw + w^2 = m^2 - PF^2 \qquad (6)$$

Adding the corresponding members of equations (4) – (6), we obtain

$$3a^2 - 2a(u + v + w) + u^2 + v^2 + w^2 = l^2 + n^2 + m^2 - (PE^2 + PD^2 + PF^2)$$

so that

$$3a^2 - 2a(u + v + w) = 0$$

Solving for $u + v + w$ produces $u + v + w = \frac{3a}{2}$.

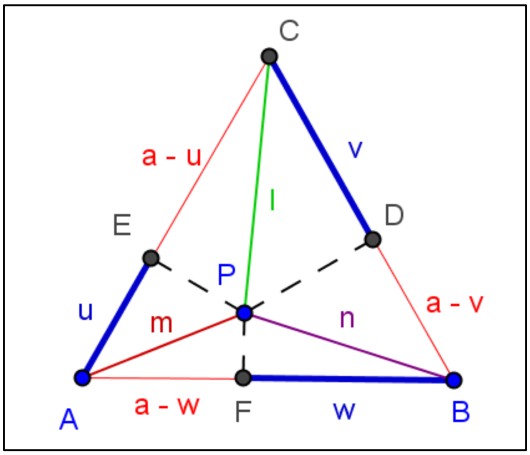


**Figure 10**: Diagram for the proof using the Pythagorean Theorem

At this point learners might wonder whether we can provide a more elegant proof, a proof that would allow us to "see" better why AE + CD + BF = $\frac{3a}{2}$. One of such proofs is the following:

From every vertex of the triangle construct a perpendicular line to one of the two sides of the equilateral triangle as shown in figure 11. Thus, $\overleftrightarrow{AC_1} \perp \overline{AB}$, $\overleftrightarrow{CB_1} \perp \overline{AC}$, and $\overleftrightarrow{BA_1} \perp \overline{BC}$. Since $\angle \mathrm{BA}C_1$ is a right angle and $m(\angle \mathrm{BAC}) = 60°$, we conclude that $m(\angle \mathrm{CA}C_1) = 30°$. Because $\angle \mathrm{AC}C_1$ is also a right angle, we know that $m(\angle A_1C_1B_1) = 60°$. Similarly, $m(\angle C_1B_1A_1) = m(\angle B_1A_1C_1) = 60°$. Thus, $\Delta A_1B_1C_1$ is equilateral. From P construct perpendicular segments to the sides of $\Delta A_1B_1C_1$. We infer that EC = PG since ECGP is a rectangle. Analogically, DB = PH and FA = PI. Thus, EC + DB + FA = PG +PH + PI = $h$, the measure of the altitude of $\Delta A_1B_1C_1$. But $h = \frac{\sqrt{3}s}{2}$ where $s$ is the measure of the side of $\Delta A_1B_1C_1$. Now, $s = A_1A + \mathrm{A}C_1 = \frac{a}{\sqrt{3}} + \frac{2a}{\sqrt{3}} = \frac{3a}{\sqrt{3}}$. Finally, $h = \frac{\sqrt{3}s}{2} = \frac{\sqrt{3}}{2}\frac{3a}{\sqrt{3}} = \frac{3a}{2}$. Therefore, EC + DB + FA = $\frac{3a}{2}$.

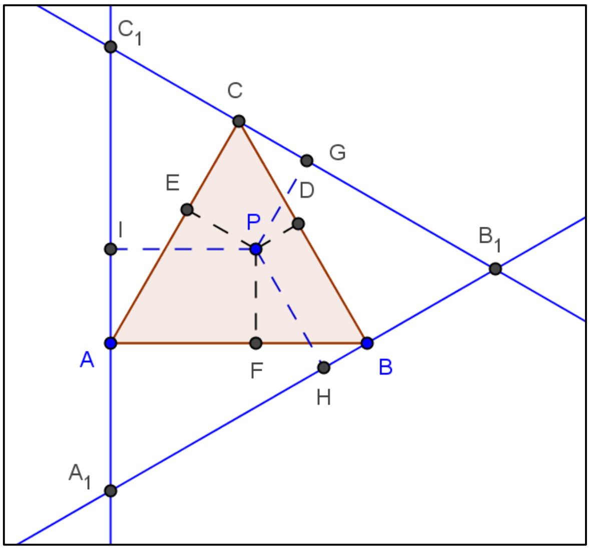


**Figure 11**: Diagram for an elegant proof of EC + DB + FA = $\frac{3a}{2}$

## 5. Extending Viviani's theorem to interior partial areas of an equilateral triangle

One of the powerful features of GG is that it often allows the user to visualize possible relationships and investigate effortlessly "what-if" questions. To this end, and prompted by the pattern involving specific segments discussed in the previous section, the inquisitive learner may wonder whether there is a relationship between the sum of the areas of the shaded triangles and the area of the equilateral triangle (figure 12).

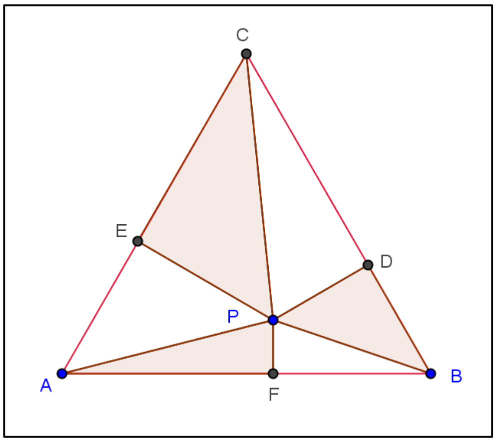


**Figure 12**: Is there a relationship between the shaded area and the area of ΔABC?

Some students may visualize and conjecture that the relationship between the shaded areas and the area of the triangle seems to be 1:2. Then they can use GG to quickly verify or refute their conjecture. Figure 13, and further dragging of point P, suggests that the conjecture is plausible. Notice that in figure 13 SumofAreas represents the sum of the three shaded areas while AreaABCover2 represents half of the area of the equilateral triangle.

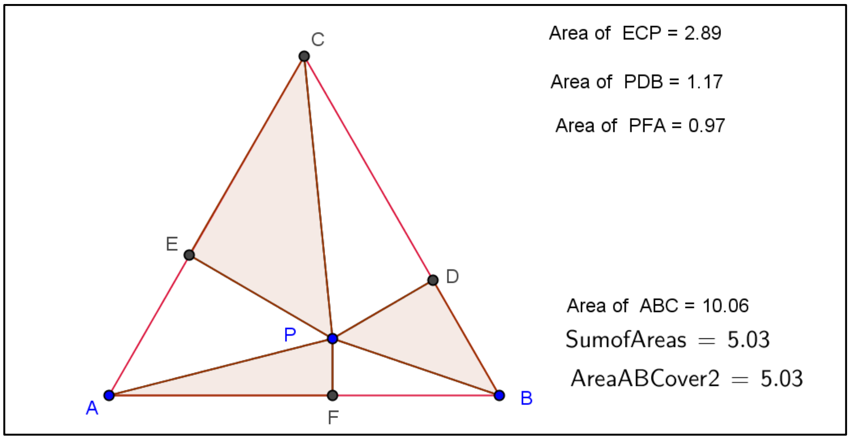


**Figure 13**: The shaded area equals the non-shaded area

A proof showing that Area(ΔAPF) + Area(ΔCEP) and Area(ΔBDP) = $\frac{Area\Delta ABC}{2}$ follows:

Through P, construct parallel segments to each of the sides of the triangle ABC. Thus, $\overline{KH} \parallel \overline{AB}$, $\overline{IL} \parallel \overline{BC}$, and $\overline{JG} \parallel \overline{AC}$ (figure 14). Let's consider first quadrilateral AIPK, which is portioned into a parallelogram AJPK and an equilateral triangle JIP. Since a diagonal divides a parallelogram into two congruent triangles, we have that Area(ΔAPK) = Area(ΔPAJ). Because an altitude divides an equilateral triangle into two congruent triangles, Area(ΔJPF) = Area(ΔIPF). Therefore, Area(ΔAPF) = Area(ΔAPK) + Area(ΔIPF). Similarly, Area(ΔBPD) = Area(ΔBPI) + Area(ΔGPD) and Area(ΔCPE) = Area(ΔCPG) + Area(ΔKPE). In other words, the shaded area equals the non-shaded area.

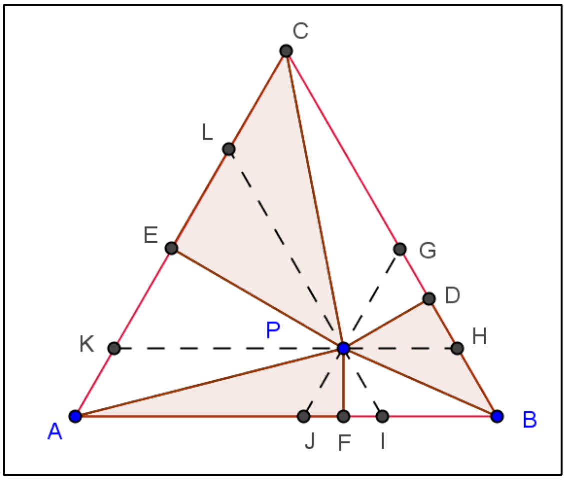


**Figure 14**: Diagram to prove that the shaded area equals the non-shaded area

## 6. Concluding remarks

In this article I have outlined how we can use GeoGebra to guide learners to discover and extend one of the most beautiful and elegant theorems of modern Euclidean geometry. In this process, students experience a glimpse of real mathematics by making discoveries, formulating conjectures, and developing mathematical arguments, which are all activities that are at the very core of doing mathematics. I challenge the reader to think of other polygons to which we can extend any of the previous results. In this way, the reader may experience the thrill of discovering new conjectures and theorems, at least new to him or her.

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