

# Kinematic Geometry

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**ABSTRACT:** This contribution deals with the applications of dynamic system GeoGebra in the teaching and learning of geometry, especially kinematic geometry. Our aim is to increase the interest of students in studying geometry at secondary schools and mainly at colleges. One possible approach of improvement in studying geometry is the integration of computer software in teaching process. In this article we will demonstrate the advantages of dynamic geometry system based on examples from the field of kinematic geometry in the plane. We will introduce main definitions and theorems and concentrate on creation of planar curves by special motions of geometric objects. The usage of software GeoGebra will be shown using examples.

**KEYWORDS:** kinematic geometry, classical geometry, motion

## 1 Introduction

This paper addresses the applications of dynamic system GeoGebra in teaching and learning geometry. I have my own experience of teaching classical, descriptive, kinematic, and computational geometry at universities – Charles University in Prague – Faculty of Mathematics and Physics and Czech Technical University in Prague – Faculty of Architecture. College mathematics and geometry is very difficult for many students. It is necessary to motivate and to arouse their interest in geometry. One possible approach of improvement in studying geometry is the integration of computer software in teaching process. This way seems to be interesting, attractive and motivational for students. Indeed the usage of computers in education is very current.

In my lessons I use computer software for visualization, for the proving geometric problems in the plane and in the space, for the demonstration of the application of geometry in practice, for the creations of study materials for home schooling and e-learning or for the transformation geometric problems into mathematical form. I work with various computer software and of course with GeoGebra. Nowadays we have extensive database of geometric tasks, images and 3D models – the outputs of these software. You can see web pages <http://surynkova.info> [Sur12].

Here we will demonstrate the advantages of dynamic geometry system on examples from the field of kinematic geometry in the plane. For further understanding it is necessary to explain the theoretical part of kinematic geometry in the plane. We will introduce briefly main parts of kinematic geometry and concentrate on creation of planar curves by special motions of geometric objects. The usage of software will be shown on some concrete examples. We will describe determinations of special motions in GeoGebra and show the usage of dynamic tools in this software on concrete examples. We will focus on geometric motion determined by envelopes and paths and motion determined by fixed and moving centrodes.

For the creation of the outputs and images we use GeoGebra. We also have web pages (see <http://surynkova.info/geogebra.php>) with database of examples of motions in the plane. We provide the access to this database to our students, so students have dynamic worksheets at their disposal.

The rest of this paper is organized as follows: the next section (Kinematic geometry) is devoted to the theoretical part of kinematic geometry and to some special motions in the plane. All illustrations are made in GeoGebra system. In conclusion we will discuss the advantages of GeoGebra and the responses from our undergraduate students.

## **2 Kinematic geometry**

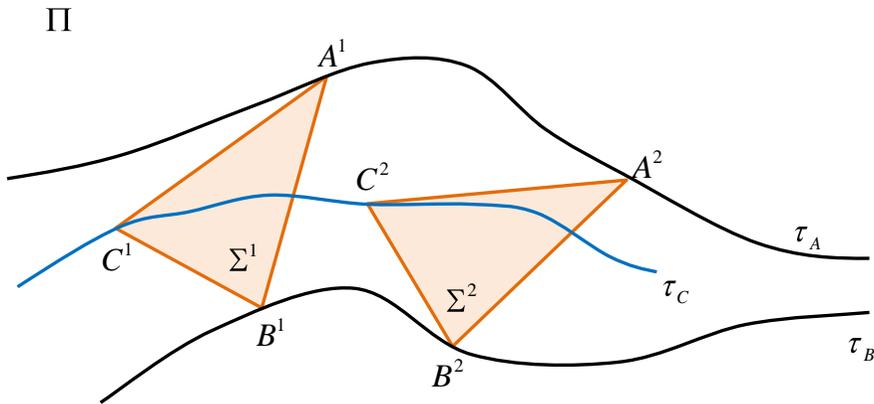
Kinematic geometry in the plane is a branch of geometry which deals with the geometric properties of objects which are created by motion of moving plane. Geometric motions – that is without regard to the cause of the motion, velocity and acceleration. Theoretical kinematics is a large subject and it is not possible to treat it completely in this contribution. We will restrict to essential basics.

We consider unbounded infinite plane which contains geometric elements (points and curves). We mainly treat such elements as points, straight lines, circles and line segments and study the geometric properties

which arise as the move in the plane. Especially we are dealing with aspects of transformation geometry. We consider those transformations in the Euclidean plane such that all distances remain fixed during the motion.

## 2.1 The determinations of geometric motion

Let  $\Sigma$  be *the moving plane* which slide over *the fixed plane*  $\Pi$ . The moving plane  $\Sigma$  contains curves and points which are at each instant of the motion identical. In the fixed plane there are generated the roulettes and the envelopes. Any point of the moving plane  $\Sigma$  describes a curve, its *path* (often called *roulette*), in the fixed plane  $\Pi$ . Any curve of the moving plane  $\Sigma$  describes a curve, its *envelope*, in the fixed plane  $\Pi$ . The path is the locus of a point in the moving plane. Geometrical envelope of a family of curves in the moving plane is a curve which at each of its points is tangent to a curve of the family. Let  $\Sigma^1, \Sigma^2, \Sigma^3, \dots$  denote the sequence of positions of the moving plane  $\Sigma$ . The positions in the moving plane of points  $A, B, C, \dots$  will be  $A^1, B^1, C^1, \dots$  when  $\Sigma$  is at  $\Sigma^1$  and  $A^2, B^2, C^2, \dots$  when  $\Sigma$  is at  $\Sigma^2$ . It is analogical for curves. Figure 1 shows an example of the motion of the moving plane.

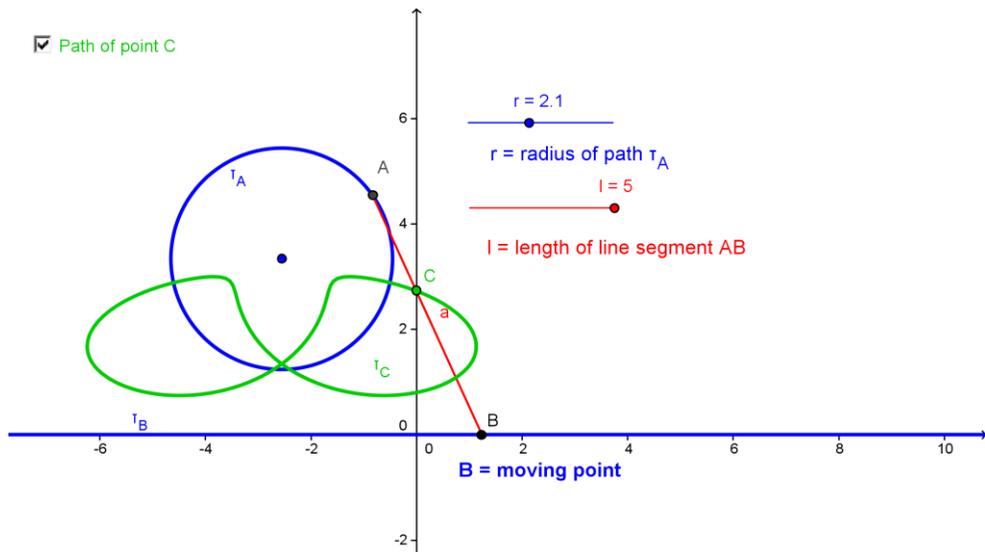


**Figure 1:** An illustration of the moving plane  $\Sigma$  containing points  $A, B, C$  (with line segments) which slides over the fixed plane  $\Pi$ . Points  $A, B, C$  describe curves  $\tau_A, \tau_B, \tau_C$  - the paths. The paths  $\tau_A, \tau_B$  are given, the path  $\tau_C$  is obtained by moving. Indices denote positions of the moving plane. All distances remain fixed.

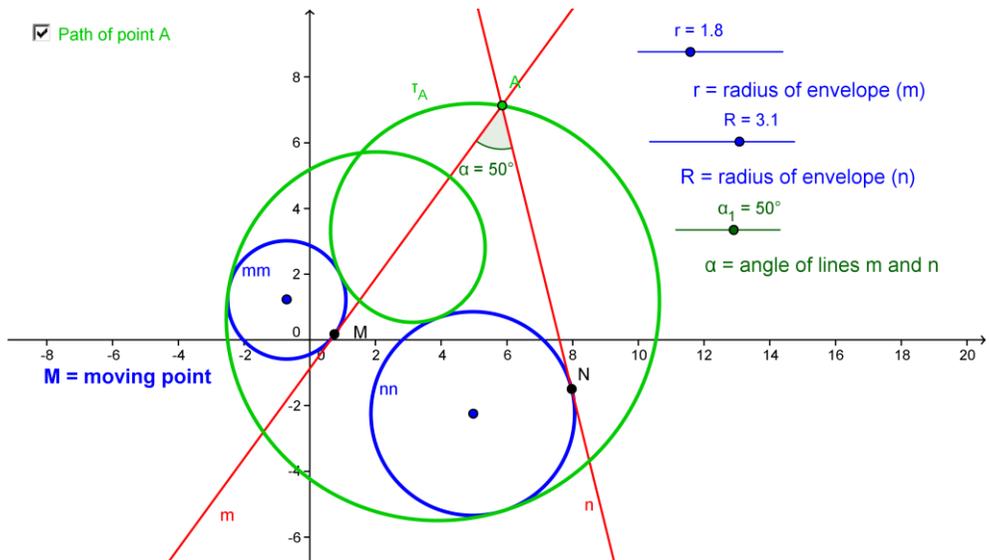
Let us discuss the determination of the motion in the plane. There are several possibilities how to define the motion:

- The motion is completely determined by paths  $\tau_A$  and  $\tau_B$  of two points  $A$  and  $B$  (end points of segment line).** These points are in the moving plane, the paths are in the fixed plane. The following equations are satisfied:  $|A^1B^1| = |A^2B^2| = |A^3B^3| = \dots$  See figure 2.
- The motion is completely determined by envelopes ( $m$ ) and ( $n$ ) of two curves  $m$  and  $n$ .** These curves are in the moving plane, the envelopes are in the fixed plane. The following equations are satisfied:  $|\square m^1n^1| = |\square m^2n^2| = |\square m^3n^3| = \dots$  See figure 3.
- The motion is completely determined by envelope ( $m$ ) of curve  $m$  and path  $\tau_A$  of point  $A$ .** The following equations are satisfied:  $|A^1m^1| = |A^2m^2| = |A^3m^3| = \dots$  See figure 4.

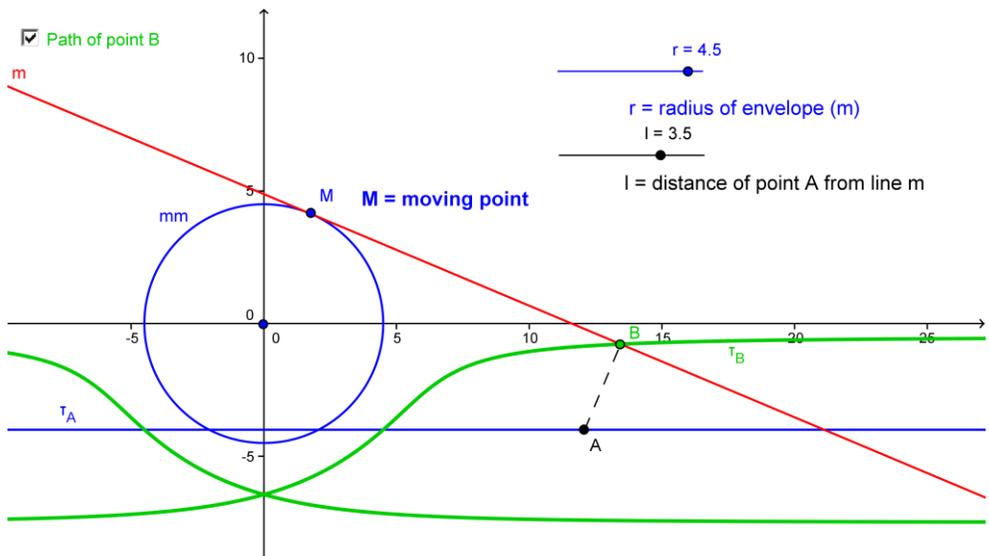
In special cases the envelope of a family of curves in the moving plane degenerates into point. These situations are shown in figures 5 and 6. The proofs of these theorems can be found in [BR79] and [Lock67].



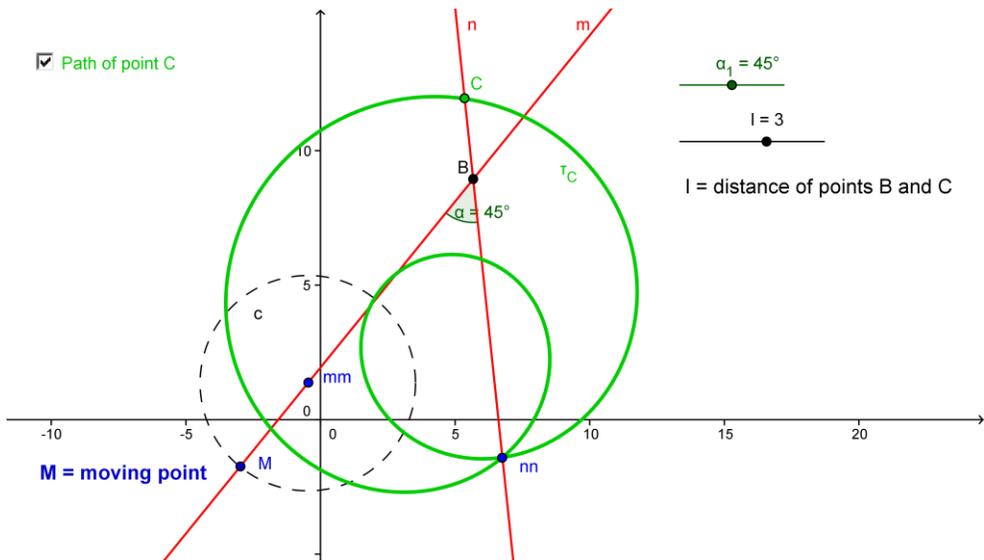
**Figure 2:** The motion is given by the paths  $\tau_A$  and  $\tau_B$  (here circle and line) of two points  $A$  and  $B$ . The path  $\tau_C$  of point  $C$  is obtained by moving. We can change some parameters in GeoGebra worksheet.



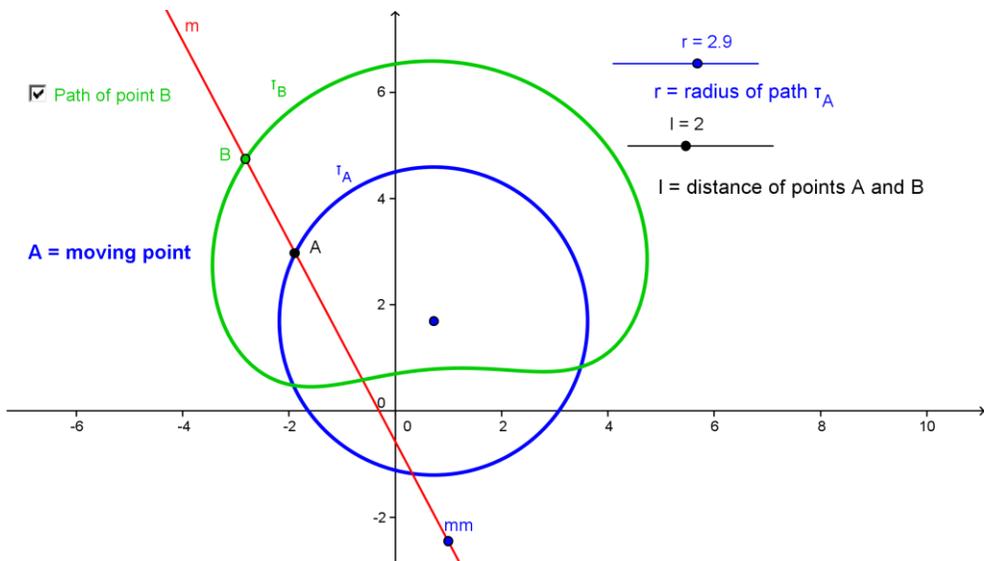
**Figure 3:** The motion is given by the envelopes ( $m$ ) and ( $n$ ) (here two circles) of two curves  $m$  and  $n$  (here two lines). The path  $\tau_A$  of point  $A$  is obtained by moving. We can change some parameters in GeoGebra worksheet.



**Figure 4:** The motion is given by the envelope ( $m$ ) (here circle) of curve  $m$  (here line) and the path  $\tau_A$  (here line) of point  $A$ . The path  $\tau_B$  of point  $B$  is obtained by moving. We can change some parameters in GeoGebra worksheet.



**Figure 5:** The motion is given by the envelopes ( $m$ ) and ( $n$ ) of two curves  $m$  and  $n$  (here lines) which degenerate into points. The path  $\tau_C$  of point  $C$  is obtained by moving. We can change some parameters in GeoGebra worksheet.



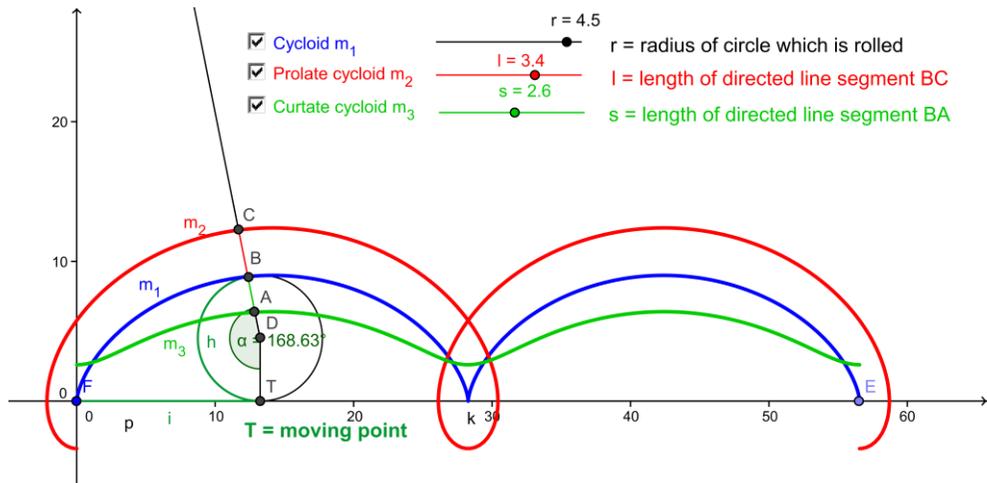
**Figure 6:** The motion is given by the path  $\tau_A$  (here circle) of point  $A$  and the envelope ( $m$ ) of curve  $m$  (here line) which degenerate into points. The path  $\tau_B$  of point  $B$  is obtained by moving. We can change some parameters in GeoGebra worksheet.

There is one more possibility how to define the motion. A more detailed study shows that the motion is completely determined by the form of the two centrodes - *the fixed centrode* of the motion in the fixed plane and *the moving centrode* in the moving plane. The centrodes are special curves which are tangent to one another at each instant of the motion and there is no slipping during the motion. The motion is obtained by rolling (without slipping) the moving centrode in the moving plane on the fixed centrode in the fixed plane.

We may also regard the plane  $\Sigma$  which we had considered movable as fixed and the plane  $\Pi$  which we had considered fixed as moveable. That is we may interchange the roles of the two planes. This motion is called *the inverse motion*. The original motion is called *the direct motion*. One motion determines the other one and the inverse of the inverse motion is the direct motion. It is possible to prove that with interchanging the roles of the centrodes we get also the *inverse motion*.

From the fact that the centrodes roll on each other without slipping it follows that the arc bounded by any two points on the fixed centrode has the same length as the arc bounded by the corresponding points on the moving centrode.

**d) The motion is completely determined by the fixed centrode  $p$  and the moving centrode  $p$ . See figure 7.**



**Figure 7:** The motion is given by the fixed centrode  $p$  (here line) and the moving centrode  $h$  (here circle). The paths  $\tau_A, \tau_B, \tau_C$  of points  $A, B, C$  are obtained by moving. These paths are called cycloids. We can change some parameters in GeoGebra worksheet.

## 2.2 Special motions in the plane

As was mentioned above a plane motion may be defined by its centrodes. The simplest example is the case where both centrodes are circles or one centrode is circle and the second is straight line. These motions are called *cyclical motions*. The motions are classified according to the type and relative positions of the centrodes. The example where a circle is rolled on a straight line was illustrated in figure 7. That is the moving centrode is the circle and the fixed centrode is the straight line. The paths of points which are obtained by rolling the circle on the straight line are called *cycloids*, *curtate cycloids* or *prolate cycloids*, generally *cycloid*. The motion is called *cycloidal*. The cycloid is the path of a point on the circumference of the rolling circle, the prolate cycloid is the path of a point outside the rolling circle and the curtate cycloid is the path of a point inside the rolling circle. The parametric equations of the cycloid are  $[x = r\alpha - v\sin \alpha, y = r - v\cos \alpha]$  where  $r$  is the radius of the rolling circle,  $\alpha$  is the angle at which the rolling circle has rotated (real parameter), in this example within the interval of  $\langle 0, 4\pi \rangle$  and  $v$  is distance between the center of the rolling circle and a point on the path. If  $v = r$ , we get the cycloid, if  $v > r$ , we get the prolate cycloid and if  $v < r$ , we get the curtate cycloid. Applications of cycloids are very important. More details can be found in [Ru00] and [Lock67].

We can interchange the roles of the two centrodes. That is a straight line is rolled on a circle. The moving centrode is the straight line and the fixed centrode is the circle. The paths of points which are obtained by rolling the straight line on a circle are called *the involute of the circle*. The motion is called *involute*. The classification of paths is similar to the cycloidal motion. The paths of points which are obtained by rolling the straight line on the circle are called *involutives*, *curtate involutes* or *prolate involutes*.

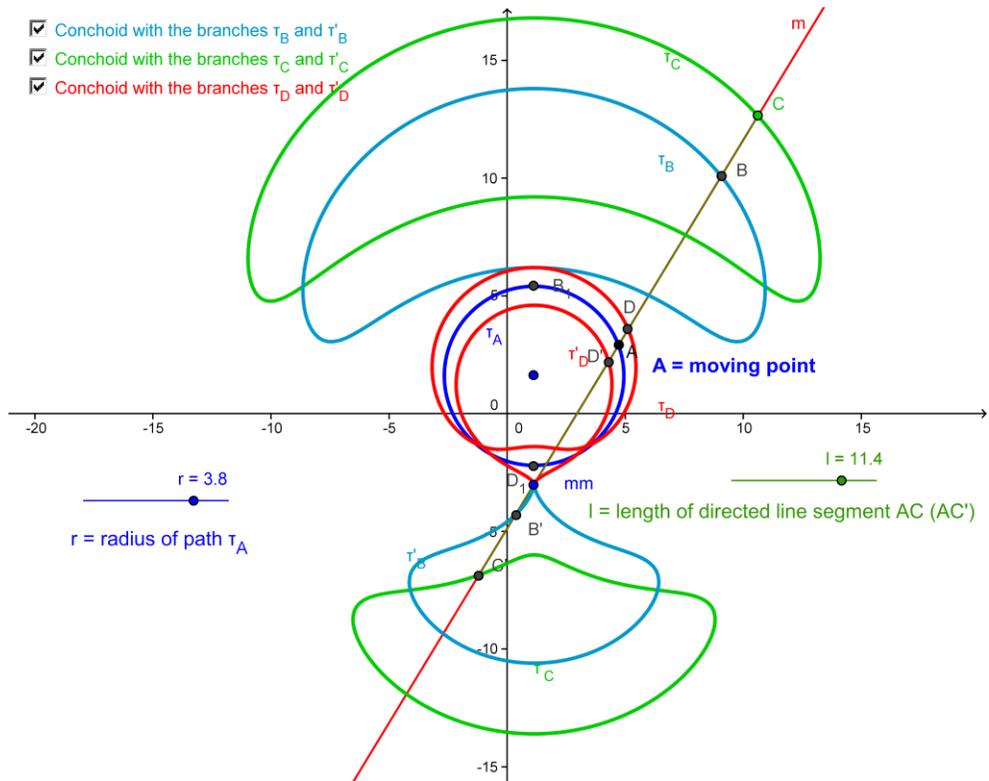
If we consider the example where a circle is rolled on a second circle there are three cases. The paths of points which are obtained by rolling the circle on the outside of the fixed circle are called *epicycloids*. The motion is called *epicycloidal*. The epicycloid is the path of a point on the circumference of the rolling circle, the prolate epicycloid is the path of a point outside the rolling circle and the curtate epicycloid is the path of a point inside the rolling circle.

The paths of points which are obtained by rolling the circle in the fixed circle are called *hypocycloids*. The radii of the two circles cannot be equal. The motion is called *hypocycloidal*. The hypocycloid is the path of a point on the circumference of the rolling circle, the prolate hypocycloid is the

path of a point outside the rolling circle and the curtate hypocycloid is the path of a point inside the rolling circle.

The paths of points which are obtained by rolling the circle on the outside of the fixed circle which is inside the rolling circle are called *pericycloids*. These curves are same as epicycloids. The motion is called *pericycloidal*.

A *conchoid* is a curve derived from a fixed point  $O$ , another curve, and a length  $d$ . For every line through  $O$  that intersects the given curve at  $A$  the two points on the line which are of  $d$  distance from  $A$  are on the conchoid. We can get the branches of this curve by the conchoid motion which is given by path  $\tau_A$  of point  $A$  and envelope ( $m$ ) of straight line  $m$  which degenerate into point. Envelope ( $m$ ) is the fixed point  $O$ . See figure 8, there are some examples of various paths.



**Figure 8:** Conchoid motion where the path  $\tau_A$  of point  $A$  is the circle. We can change some parameters in GeoGebra worksheet.

There exists a wide range of various special motions in the plane. We mentioned only the most important and the best known. We can also study the properties of curves; we refer the reader to [AGS06] and [PAHK07].

### 3 Conclusion

We discussed possible approaches how to increase the interest of students in studying classical geometry at secondary schools and colleges. The integration of computer software, mainly GeoGebra, in the teaching process we demonstrated on the examples from the field of kinematic geometry.

In our lessons we also show special constructions applied in descriptive geometry and due to included functions and tools students can discover proofs more easily. Of course, students also create some examples and tasks themselves. We have web pages with database of geometric tasks in the plane and in the space.

The feedbacks from students using GeoGebra in learning geometry are very positive. Students are satisfied because GeoGebra is very useful for them. Kinematic geometry in the plane is more understandable and geometry in general becomes modern discipline. The outputs from GeoGebra can also be used for e-learning not only for our students.

In future work we will focus on further methods which can improve the teaching process. We plan to extend our gallery of geometric tasks.

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