

Visualizing complex integration with *GeoGebra*

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ABSTRACT: In this paper we will explore ways in which complex integrals can be constructed geometrically using *GeoGebra*. We will try to deduce laws of complex integration by experimental means.

KEYWORDS: *GeoGebra*, complex integration, complex number, general residue theorem, Cauchy formula.

1 Introduction

There exists a common misconception that complex integration is too harsh to be visualized. Still we can find some geometrical representations [WE12, LAN99, NEE98, AH10]. As it turns out complex integral can be visualized as path [LAN99] or be constructed using vectors [AH10], there are also other ideas of visualizing complex integral [WE12]. In “Visual Complex Analysis” Tristan Needham did tremendous work on this subject trying to show that complex integration is just continuation of real integral and can be produced in a plane using complex numbers, vectors and sums [NEE98]. In this paper we will follow Tristan Needham's ideas and expand them to suit *GeoGebra* (more information about dynamical geometry software *GeoGebra*: <http://www.geogebra.org>).

Riemann integral (usually called real integral) is usually calculated following idea of Riemann sum or trapezoidal rule, but idea in both cases is the same - adding up function value at the point in interval multiplied by size of interval as this mentioned interval length goes to zero [PU10]. This

idea geometrically is visualized as area under the curve. Formally, let $a = x_0 < x_1 < \dots < x_n = b$, then we get

$$\int f(x)dx = \lim \sum f(t_k)\Delta x_k, \quad t_k \in [x_{k-1}, x_k]$$

We will try to extend this idea to complex plane. First we will choose two points in \mathbb{C} and then connect them, then chop this path to small intervals and find function value obtained in this interval. Multiplying obtained intervals and values and summing them up we should obtain complex integral value. Formally we have such plan:

1. Create points $a \in \mathbb{C}$ and $b \in \mathbb{C}$.
2. Create path between a and b , then chop it to small pieces Δ_i .
3. Find points z_i inside obtained intervals.
4. Calculate function value at each point $f(z_i)$.
5. Calculate sum $\sum f(z_i)\Delta_i$.

2 Constructing complex integral

Let us evaluate integral $\int e^z dz$. We get into difficulties right away, because in complex plane there are infinite ways we can connect two points. For our first graph let a path between these two points be a straight line. We are forced to use vectors to represent this path.

In this paper we will use *Sequence* command in *GeoGebra* to create lists of elements, but *Spreadsheet* can be used to make same constructions. All constructions used in this paper and more can be found at: <https://sites.google.com/site/visualcomplexanalysis/>.

1. Let $a \in \mathbb{C}$ and $b \in \mathbb{C}$.

```
>>> a=0+i
```

```
>>> b=1+0*i
```

2. We create vectors Δ_i . In this case we will create 10 equal vectors from point a to b .

```
>>> Sequence[Vector[a+n*Vector[a,b]/10, a+(n+1)*Vector[a,b]/10],
n, 0, 9]
```

3. Now we need a list of points in the middle of these vectors, formally called z_i .

```
>>> Sequence[a+n*Vector[a,b]/10+Vector[a,b]/20, n, 0, 9]
```

4. Now we calculate values at points in list2, thus obtaining $f(z_i) = e^{z_i}$. Later we will be able to experiment with different functions.

```
>>> Sequence[e^Element[list2,n], n, 1, 10]
```

5. We will multiply obtained values by representative vectors obtaining $f(z_i)\Delta_i$ and then add these vectors to get value of our integral, which is equal to $\sum f(z_i)\Delta_i$.

```
>>> Sequence[(Element[list1, n]+0*i) Element[list3, n], n, 1, 10]
```

```
>>> Sequence[Sum[Sequence[Element[list4,n],n,1,m]]+0*i,m,1,10]
```

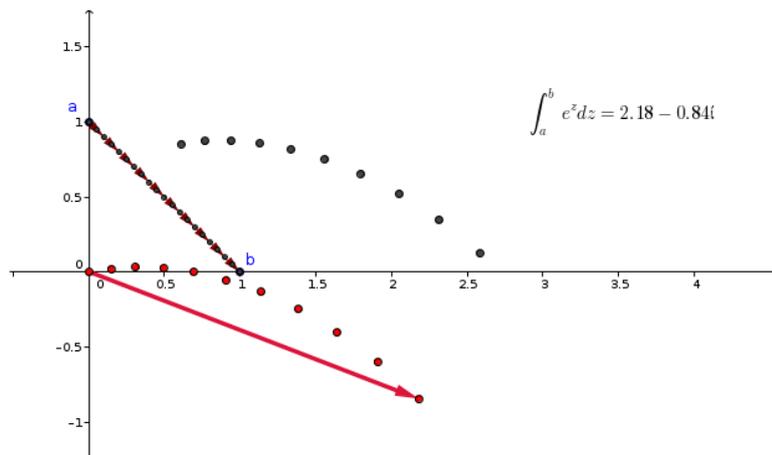


Fig. 1. Visual representation of $\int e^z dz$.

Obtained sum (see Fig. 1) with each step gets closer to point $2.18 - 0.84i$, which is actually approximately equal to actual integral value. To get more accurate result we can make Δ_i variable and decrease its size till we get desired accuracy. From real case it is good guess that complex integrals could obey same integration laws as real ones and $\int e^z dz = e^b - e^a$.

We can visually make sure that this is true by adding a new point:

```
>>> e^b-e^a
```

Now by dragging point a to the position of point b and point a to position of point b we are changing integration order. It is easy to see that in such case we obtain $-2.18 + 0.84i$, thus the sign of the function has changed. Analogous to real case we just witnessed the property of complex

integration $\int_a^b f(z) dz = -\int_b^a f(z) dz$.

In the same way it is possible to convince yourself with other complex integration properties, see [INT84] or any other introductory course book to complex analysis.

It is easy to change integration path to arc (see Fig. 2) or even random path (we will leave this as exercise to reader). Still in each case we will get same result equal to $e^b - e^a$. Hence we are forced to come up with idea that changing integration path will not affect integral value. For formal proof look at [AH10].

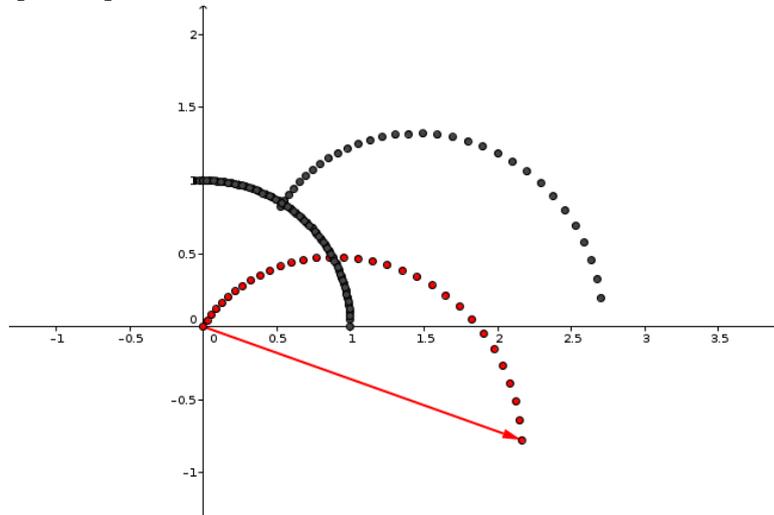


Fig. 2. Visual representation of $\int e^z dz$. Arc path.

2 Closed circular path

While calculating real integral choosing start point and end point to be equal was impossible task. In complex plane we can make same point to be starting point and end point and define path from start to finish. For simplicity we will create circular path, as we saw already circular path can be deformed and this will not affect integral value. We will explore integral $\oint 1/z dz$.

It is easy to create circular path in *GeoGebra* using polar coordinate form, all other steps are similar to previous example.

```
>>> Sequence[Vector[(1;n pi/18),(1;(n+1) pi/18)],n,0,35]
>>> Sequence[(1;n pi/18+pi/36)+0*i,n,0,35]
>>> Sequence[1/Element[list2,n], n, 1, 36]
>>> Sequence[(Element[list1, n]+0*i) Element[list3, n], n, 1, 36]
>>> Sequence[Sum[Sequence[Element[list4,n],n,1,m]]+0*i,m,1,36]
```

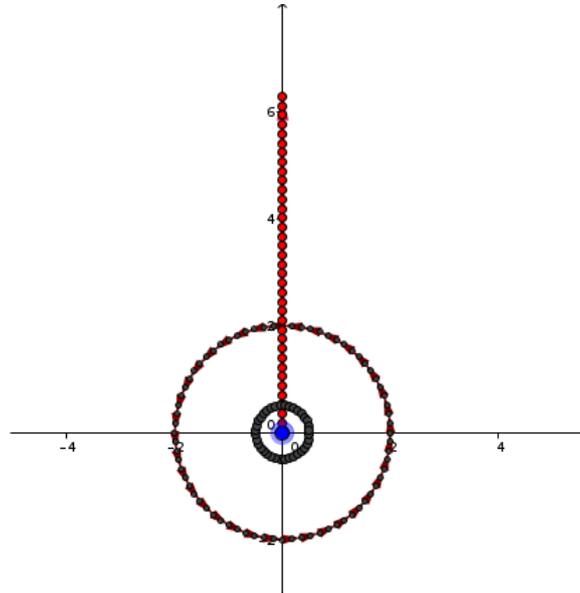


Fig. 3. Visual representation of $\oint 1/z dz$.

We just obtained astounding result $\oint 1/z dz = 2\pi i$ (see Fig. 3). As it turns out it has something to do with $z \neq 0$. We will call this point singularity and explore this phenomenon further.

3 Exploring General residue theorem

We will try to evaluate $\oint 1/(z-2i) dz$. This function has singularity at $z = 2i$, thus again as in previous example we expect something strange to happen as we encircle this point. In *GeoGebra* we create construction with movable integration path to make it easier to see how changing integration circle position will affect integral value.

```

>>> b=0+0*i
>>> Sequence[Vector[b+(1;n pi/18),b+(1;(n+1) pi/18)],n,0,35]
>>> Sequence[b+(1;n pi/18+pi/36)+0*i,n,0,35]
>>> Sequence[1/(Element[list2,n]-2*i), n, 1, 36]
>>> Sequence[(Element[list1, n]+0*i) Element[list3, n], n, 1, 36]
>>> Sequence[Sum[Sequence[Element[list4,n],n,1,m]]+0*i,m,1,36]

```

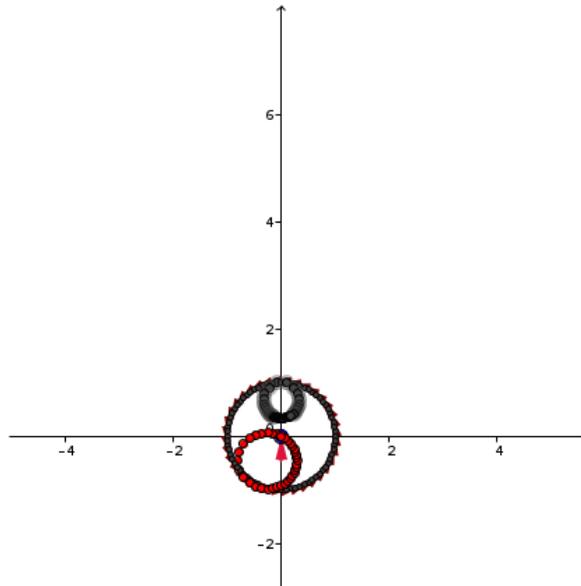


Fig. 4. Integral $\oint 1/(z-2i)dz$ near origin.

We see that near origin integral value goes to zero (see Fig. 4). From this point we can move point b and see what happens near $2\pi i$ (see Fig. 5).

We can try to change *GeoGebra* construction to encircle singularity point multiple times and see that if we encircle singularity n times we get $2\pi in$, thus let us denote number of rotations as $v(K, s_j)$. Changing function to $f(z) = 2/(z-2i)$ or even to $f(z) = z/(z-2i)$ will enable us to see that integral value depends on numerator, we will denote it $Res(f(z), s_j)$. From this point it is easy to come up with General Residue Theorem $\oint f(z)dz = 2\pi i \sum v(K, s_j) Res(f(z), s_j)$. From here reader is invited to read formal proof [AH10].

4 Exploring Cauchy formula

Following the same construction idea, from previous example, we can create graphical representation of integral $\oint z^2/(z-a)dz$ only by creating new point and redefining list3 as follows:

```
>>> a=1+i
>>> list3=Sequence[Element[list2,n]^2/(Element[list2,n]-a), n, 1, 36]
```

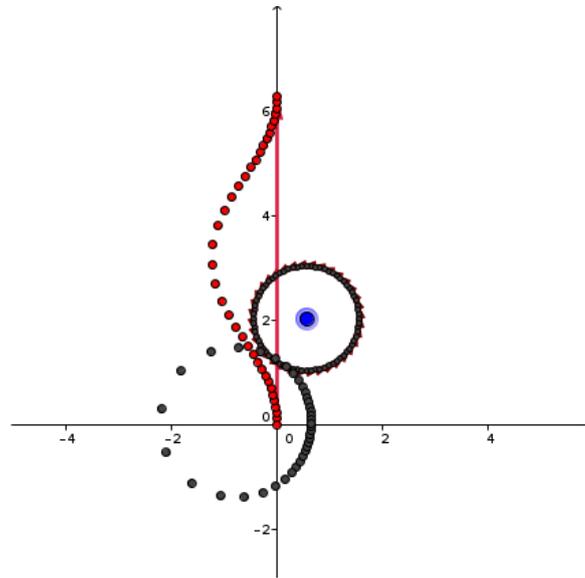


Fig. 5. Integral $\oint 1/(z - 2i) dz$ near $z = 2i$.

It is quite easy to convince yourself with truthfulness of Cauchy Formula $\oint f(z)/(z - a) dz = 2\pi i f(a)$ from General Residue Theorem, or with the help of *GeoGebra* construction we just made (see Fig. 6).

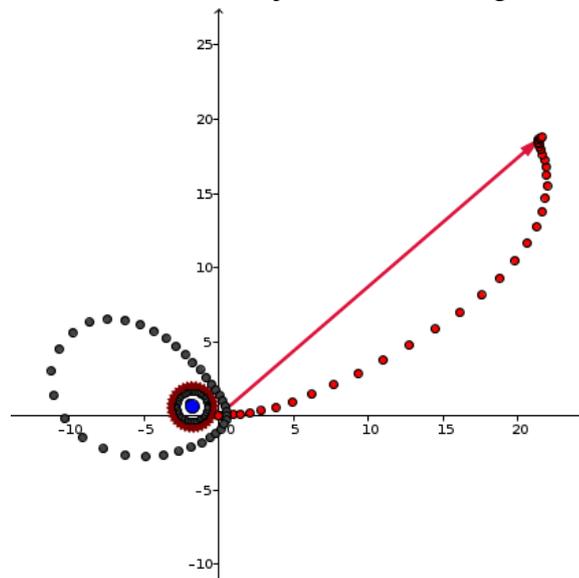


Fig. 6. Integral $\oint z^2/(z - 2i) dz$.

5 Final thoughts

Author hopes that constructions you just made will inspire to explore complex integration further. You can find a lot of beautiful problems in Tristan Needham “Visual Complex Analysis” [NEE98]. There are few ideas what to do next: compute other integrals and construct different integration paths; explore what happens if you encircle singularity n times; explore what happens if integral has couple singularities and you encircle just some of them.

Geometrical reasoning this time was not strict enough to hold it as proof on its own, it was given following hope that this will enable interested reader to understand properties of complex integration in more vivid geometrical manner. It is strongly recommended to read more about complex integration [NEE98] and then come back to *GeoGebra*.

More information about visualizing complex variable and constructing complex integrals with help of *Spreadsheet* in *GeoGebra* can be found at <https://sites.google.com/site/visualcomplexanalysis/>.

References

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